EXCERPTS FROM

Full Frontal Calculus

The full book is available as a paperback at Amazon.com and as pdf at BraverNewMath.com

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Cover photo by Shannon Michael.
(In Latin, *calculus* = “small stone”.)

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ONLY THESE sections are contained in "EXCERPTS from FFC"

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Preface for Teachers

"Anyone who adds to the plethora of introductory calculus textbooks owes
an explanation, if not an apology, to the mathematical community."
-Morris Kline, *Calculus: An Intuitive and Physical Approach*

Full Frontal Calculus covers the standard topics in single-variable calculus, but in a somewhat unusual way. Most notably, in developing calculus, I favor infinitesimals over limits. Other novel features of the book are its brevity, low cost, and – I hope – its style.

Why infinitesimals rather than limits? A fair question, but then, why limits and not infinitesimals? The subject's proper historical name is *the calculus **of infinitesimals***. Its basic notation refers directly to infinitesimals. Its creators made their discoveries by pursuing intuitions about infinitesimals. Most scientists and mathematicians who comfortably use calculus think about it in terms of infinitesimals. Yet in our classrooms, we hide the essence of calculus behind a limiting fig leaf, as if fearful that the sight of bare infinitesimals could make fragile young ladies faint at their desks. Please. Spare us your smelling salts. The tightly corseted pre-Robinsonian era ended half a century ago.

Infinitesimals can now be made every bit as rigorous as limits, though full rigor is beside the point in freshman calculus, the goals of which are deep intuition and computational facility, each of which we enhance by admitting infinitesimals into our textbooks. By using infinitesimals, we help our students relive the insights of the giants who forged the calculus, rather than those of the janitors who tidied up the giants' workshop.

The book is short because it need not be long. If fools like you and me (and Silvanus Thompson) can master the calculus, it cannot be so complex as to require a 1000-page instruction manual.

I have chosen self-publication primarily because my experiences with a conventional publisher for my first book, *Lobachevski Illuminated*, were not altogether happy. Despite winning the Mathematical Association of America's Beckenbach Prize in 2015, that book passed most of its first decade in awkward electronic limbo – until 2021, when the American Mathematical Society gave it its first proper print run. By retaining control over the present book's publication, I can ensure that it remains available and inexpensive to my students (and yours).

For the past several years, while using drafts of *Full Frontal Calculus* in my classes at South Puget Sound Community College, I've offered students extra credit points for catching typos. Their quarry, once plentiful, has now been satisfactorily reduced. Any remaining typos or errors are, of course, due solely to my students' appalling negligence. Please let me know of any surviving mistakes that you might notice. (You can find my contact information at my website, BraverNewMath.com.) And if, after reading the book, you find yourself clamoring for full frontal *multivariable* calculus, let me know that, too. Given sufficient demand, who knows, I might even write it.

Preface for Students

"Allez en avant, et la foi vous viendra."
-Jean Le Rond D'Alembert

Fair warning: This book's approach to calculus is slightly unorthodox, particularly in its first two chapters. Consequently, you are unlikely to find other books or videos that will show you how to solve many of its homework problems.[*] This, despite what you might initially think, is a good thing; it will force you, from the beginning, to become an active reader – and thus to become more in tellectually self-reliant.

Full Frontal Calculus is meant to be read slowly and carefully. Ideally, you should read the relevant sections in the text before your teacher lectures on them. The lectures will then reinforce what you've understood, and clarify what you haven't. Read with pencil and paper at the ready.[†] When I omit algebraic details, you should supply them. When I use a phrase such as "as you should verify", I am not being facetious. Only *after* reading a section should you attempt to solve the problems with which it concludes. When you encounter something in the text that you do not understand (even something as small as an individual algebraic step), you should mark the relevant passage and try to clear it up, which may involve discussing it with your classmates or teacher or reviewing prerequisite material. Regarding this last point, calculus students commonly find that their grasp of precalculus mathematics is weaker than they had supposed. Whenever this happens ("I've forgotten how to complete the square!"), do not despair, but *do* review the relevant topics.[‡] Be assiduous about repairing foundational cracks whenever you discover them, for the machinery of calculus is heavy; attempts to erect it on porous precalculus foundations end poorly.

As with learning a language or a musical instrument, learning calculus requires tenacity of purpose: To succeed, you must devote several hours a day to it, day after day, week after week, for many months. Fortunately, the intrinsic rewards in learning calculus – as with a language or the violin – are substantial. And, of course, knowing calculus extends your ability to study science, and thus eventually to enter professional scientific fields.

At the beginning, this can all seem quite intimidating. Bear in mind that tens of thousands of people succeed in learning basic calculus every year. You can be one of them. But it will require hard work, and at times, you may wonder whether it is worthwhile. It is. Go on, and faith will come to you.

Let us begin.

[*] Once we've reached cruising altitude (Chapter 3), I, your faithful pilot, will switch off the fasten-seatbelts light, and you'll be able to consult standard calculus texts and websites once again if and when you so desire.

[†] I highly recommend purchasing a print copy so that you can scrawl notes in the margins. Online reading, in my experience at least, is rarely active reading.

[‡] See this book's prequel, *Precalculus Made Difficult*, available on my website, BraverNewMath.com.

Calculus: From A to The

A calculus is a set of symbolic rules for manipulating objects of some specified type. If you've studied statistics, you've probably used the *calculus of probabilities*.* If you've studied formal logic, it follows that you've met the *propositional calculus*.† After you've mentally summed up all the little bits of knowledge in this book, you'll have learned **the calculus *of infinitesimals***.

An infinitesimal is an infinitely small number – smaller than any positive real number, yet greater than zero. Like square roots of negatives (the regrettably-named "imaginary numbers"), infinitesimals seem paradoxical: They manifestly do *not* belong to the familiar system of real numbers. Mathematicians and scientists have, nonetheless, used them for centuries, because infinitesimals help us understand *functions of real numbers* – those faithful tools with which we model, among other things, the so-called real world of experience. To gain perspective on our real homeworld, it helps to survey it from without. Such a justification for working with numbers beyond the reals should satisfy the hard-nosed pragmatist; for another, equally valid, justification, we need only observe our fellow mammals the dolphins and whales at play. There is pleasure to be had in breaching the surface of a world that normally confines us.

In biology, the naked eye is perfectly serviceable for some purposes, but a microscope's lenses reveal details of a microworld that can ultimately help explain what we experience on our familiar scale. Similarly, mathematicians have found that although the real numbers suffice for our basic measurement needs, "infinitesimal-sensitive lenses" can sharpen the pixelated image that the reals present to our naked mind's eye. Our mathematical microscope is, of course, purely mental. Cultivating a sense of what it reveals requires practice and imagination, but the essence of the idea, however, is simple: Magnitudes that appear equal to the naked eye (i.e. in terms of real numbers) may turn out, when viewed through our infinitesimal-sensitive microscope, to differ by an infinitesimal amount. Conversely, magnitudes that differ by a mere infinitesimal when viewed under the microscope correspond, when viewed with the naked eye, to *one and the same real number*.

"Interesting," you may reflect, "but I wish we'd just stick with the good old familiar real numbers." Be careful what you wish for, lest you unnecessarily limit your mathematical imagination! Yes, the reals are familiar and indispensable, but they can also be disturbingly strange; by embracing infinitesimals, we can actually divest the reals of some of their strangeness, as you'll see in the first example below. In the second example, you'll see how infinitesimals can help us bridge the qualitative divide between curves and straight lines. Bridging this divide turns out to be a major theme of calculus.

Example 1. Suppose an urn contains eight balls, one of which is red. If we draw a ball at random (so each ball is equally likely to be drawn), the probability of drawing the red one is clearly $1/8$. But suppose there are not eight, but *infinitely many* balls from which to draw. Under these circumstances, what is the probability of drawing the one red ball?

The answer seems to be "one in infinity", but what does that even mean? Well, there are two things we can definitely say about a "one in infinity" probability. The first is that it is surely *less than* "one in N" for any whole number N whatsoever. The second is that, like all probabilities, it is a

* The calculus of probabilities contains rules such as $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

† Such a calculus would contain rules such as $P \wedge Q \Longrightarrow P$.

number between 0 and 1, the values that correspond to impossibility and certainty respectively. Combining these two facts yields the following conclusion: The probability of drawing the red ball is some number that lies between 0 and 1 and is less than $1/N$ for all whole numbers N.

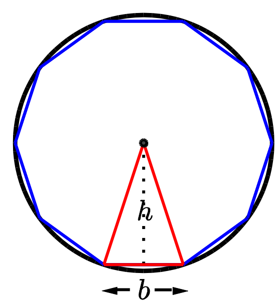
A little thought should convince you that only one *real* number satisfies both demands: **zero**. Hence, *if we confine ourselves to the real numbers*, we are forced to conclude that the probability of drawing the red ball is zero – which suggests that drawing it is not merely unlikely, but actually impossible. Moreover, the same logic applies to each one of the infinitely many balls, which leads us to the awkward conclusion that from our infinite collection, it is impossible to draw any ball whatsoever! Such is the paradoxical scene as viewed with the naked eye.

The paradox vanishes if we accept infinitesimals. For if we do, we need not conclude that the probability of drawing the red ball is zero; it could be *infinitesimal* while still meeting the two demands for a "1 in ∞" probability described above. The probability of drawing the red ball would then be unfathomably minuscule – less than $1/N$ for every whole number N, impossible to represent as a decimal, indistinguishable from zero in the *real* world – and yet, not quite zero. Consequently, the red ball *can* be drawn, though I wouldn't advise betting on it. ♦

So much for balls. Let's drop down a dimension and discuss circles. Everyone and his mother "knows" that a circle of radius r has area πr^2. But *why* is this so? Infinitesimals can help you understand.

Example 2. The area of a circle with radius r is πr^2.

Proof. Inscribe a regular n-sided polygon in the circle. Clearly, the greater n is, the closer the polygon cleaves to the circle. Even when n is relatively small (say, $n = 50$), distinguishing the two shapes is difficult for the naked eye, yet they remain distinct for any finite n. Our proof of the area formula, however, hinges on a radical reconceptualization: *We shall think of the circle as a regular polygon with infinitely many sides, each of which is infinitesimally small*. This idea will enable us to use facts about polygons (straight, simple objects) to learn about circles (curved objects).



Since a regular n-gon's area is n times that of the triangle in the figure above, its area must be $n(bh/2)$. Since $nb = P$ (the polygon's perimeter), this expression for area simplifies to $Ph/2$.

Thus the circle, *being a polygon*, has area $Ph/2$. But for our circle/∞-gon, P represents the circle's circumference (which is $2\pi r$), and h represents the circle's radius (which is r). Substituting these values into the area formula $Ph/2$, we conclude that the *circle's* area is $(2\pi r)r/2$, which simplifies to πr^2, as claimed. ■

Please dwell on this surprising, beautiful, disconcerting argument. When mathematicians began to use infinitesimals, even philosophers and theologians took note. Is a proof that uses infinitesimals a genuine proof? Is a circle really a polygon of infinitely many sides? Is it wise for finite man to reason about the infinite? Even as such philosophical debates raged (from the 17^{th} century on), mathematicians paid only halfhearted attention, busy as they were developing a potent *calculus* of infinitesimals. That this calculus worked no one questioned; that it lacked a fully comprehensible foundation no one denied. Its triumphs were astonishing. The infinitesimal calculus helped physics break free of its static Greek origins and become a dynamic modern science. And yet... all attempts to establish iron-clad logical foundations for the calculus failed. Since at least the time of Euclid (c. 300 BC), mathematics had been viewed as the archetype of logical reasoning. Small wonder then that, despite the undeniable utility of the infinitesimal calculus, its murky basis received stinging criticism. Most famously, philosopher George Berkeley suggested in 1734 that anyone who could accept the mysterious logical foundations of the infinitesimal calculus "need not, methinks, be squeamish about any point in Divinity." To compare mathematics – the traditional rock of logical certainty – to theology, nay, to assert that mathematicians, far from proceeding by perfectly rigorous thought, "submit to authority, take things on trust, and believe points inconceivable" (as Berkeley would have it) was to shake one of the very pillars of Western civilization.*

Not until the late 19th century did mathematicians discover a perfectly rigorous method (the theory of *limits*, which you'll learn about in Chapter π) to set the theorems of the infinitesimal calculus on solid foundations. That it took so long to develop these foundations is understandable, given the surprising sacrifice involved: To transfer the massive body of theorems onto the long-desired secure logical foundations, mathematicians had to sacrifice the infinitesimals themselves!

Placed firmly atop these new limit-based foundations, the many theorems of calculus that had been developed over the previous centuries were finally secure (there was no longer anything to fear from the philosophers), but the infinitesimals that had nourished the subject as it developed were ruthlessly expunged in the victory celebration. The very notion of an infinitesimal came to be viewed as an embarrassment to the brave new limit-based calculus, as though "infinitesimal" were a discredited religious idea from a more primitive time whose abandonment was necessary for the further progress of humanity. Even the subject's name was changed. What had once proudly been known as "The Calculus of Infinitesimals" thenceforth became known simply as **The Calculus**, a name whose emptiness spoke – to those, at least, with ears to hear – of the ghosts of departed quantities.†

Infinitesimal ghosts continued to haunt the calculus, for although the theory of limits had brilliantly disposed of a logical problem, it had introduced a psychological problem. In the minds of many who used calculus as a scientific tool (but who had no particular concern for the subject's esoteric logical foundations), infinitesimals remained far more intuitive than limits. For this reason, much notation that originally referred to infinitesimals was, surprisingly, retained even after the great infinitesimal purge. Naturally, the notation was reinterpreted in terms of limits, which entailed a sort of mathematical schizophrenia. One would use infinitesimal notation and think infinitesimally, but good mathematical hygiene demanded that one refrain from actually mentioning infinitesimals in public. To be sure, textbook authors would sometimes timidly advise their readers that it might be helpful to think of such and such an expression in terms of infinitesimals, but such advice was invariably followed (as if an authority figure had just returned to the room) by a stern warning that actually, infinitesimals don't exist, and one shouldn't speak about such things in polite society.

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* Berkeley was a masterful shaker of pillars. His denial that matter exists outside of minds paved the way for David Hume's philosophical demolition of causality and personal identity, and hence to Immanuel Kant's subsequent reconstruction of these ideas on a radically new philosophical basis (transcendental idealism) that he developed to refute Hume, and which then became a cornerstone of modern philosophy.

† Since "The Calculus" is only slightly more expressive than "The Thing", there was no real loss when, in time, even the definite article was shed. Hence today's unadorned *Calculus* (on tap at a college near you).

Resurrection

"I think in coming centuries it will be considered a great oddity in the history of mathematics that the first exact theory of infinitesimals was developed 300 years after the invention of the differential calculus."
- Abraham Robinson

In his landmark book *Nonstandard Analysis* (1966), from which the preceding epigraph was taken, Abraham Robinson astonished the mathematical world by using tools from 20^{th}-century logic to construct the Holy Grail of Calculus: a perfectly rigorous way to make infinitesimals logically respectable.

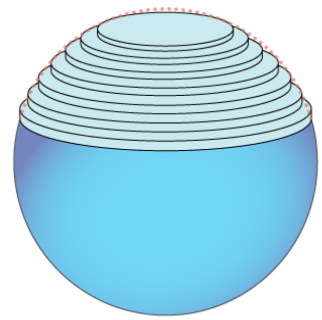
Robinson used his newly vindicated infinitesimals (which joined the familiar *real* numbers to produce an extended system of *hyperreals*) to construct an alternate foundation for calculus. His infinitesimal-based, 20^{th}-century foundation was every bit as solid as the limit-based, 19^{th}-century foundation, but the mathematical world, in the intervening years, had grown secular; its desire for the Infinitesimal Grail had been nearly extinguished. Though duly impressed by Robinson's intellectual achievement, mathematicians were largely unmoved by it, for the theory of limits (itself a century old by that time) was not only fully rigorous, but also fully entrenched. The logical foundations of calculus had long since ceased to be an active field of inquiry, and few mathematicians (who were busy, naturally, exploring other problems) cared to revisit it. Their position was quite understandable: The theory of limits, which they, their teachers, and their teachers' teachers had all mastered long ago as mere undergraduates, *worked*. From a strictly logical standpoint, it did not need to be replaced, and mathematicians are no less inclined than others to hearken unto the old saw: *If it ain't broke, don't fix it.* Thus they tended – and tend – to view Robinson's work as a remarkable curiosity. All mathematicians know of Robinson's achievement. Few have studied it in detail.

O intended reader of this book, you are not a professional mathematician. You are a student in a freshman-level class. You need not, at this point in your academic career, concern yourself with the full details of calculus's logical foundations, except to be reassured that these exist and are secure; you can study them (in either the limit-based version or the infinitesimal-based version) in the appropriate books or classes should you feel so inclined in the future. One does not take a course in Driver's Education to learn the principles of the internal combustion engine, and one does not take freshman calculus to learn the subject's deep logical underpinnings. One takes a course such as this to learn the calculus itself – to learn what it is, how to use it, and how to think in terms of it, for calculus is as much a way of thinking as it is a collection of computational tricks. For you (and for your teacher), the importance of Robinson's work lies not in its formidable logical details, but rather in the retrospective blessing it bestows upon the centuries-old tradition of *infinitesimal thinking*, a tradition that will help you understand how to think about calculus – how to recognize when calculus is an appropriate tool for a problem, how to formulate such problems in the language of calculus, how to understand *why* calculus's computational tricks work as they do. All of this becomes considerably easier when we allow ourselves the luxury of working with infinitesimals. We need no longer, as in the 1950's, blush to say "the i-word". And so, in accordance with this book's title, infinitesimals shall parade proudly through its pages, naked and unashamed.

The World of Calculus: An Overview

"I'm very good at integral and differential calculus,
I know the scientific names of beings animalculous.
In short in matters vegetable, animal, and mineral,
I am the very model of a modern major general."
- Major General Stanley, in Gilbert and Sullivan's *Pirates of Penzance*.

Calculus is traditionally divided into two branches, integral and differential calculus.



Integral calculus is about mentally decomposing something into infinitely many infinitesimal pieces; after analyzing the pieces, we then re-*integrate* them (sum them back up) to reconstitute the whole. The spirit of integral calculus hovered over our proof of the circle's area formula, when we reimagined the circle's area as the sum of the areas of infinitely many infinitesimally thin triangles. Integral calculus thus involves a special way of thinking, or even a special way of seeing. With "integral calculus eyes", one might view a solid sphere as a stack of infinitely many infinitesimally thin discs, as suggested by the figure at right. Alternately, one might imagine it as an infinite collection of concentric infinitesimally thin hollow spheres, nested like an onion's layers.

Differential calculus is about *rates of change*. An object's speed, for example, is a rate of change (the rate at which its distance from a fixed point changes in time). A bank's interest rate is a rate of change (the rate at which dollars left in a savings account change in time). Chemical reactions have rates of change (the rate at which iron rusts when left in water, for instance). The slope of a line is a rate of change (the rate at which the line rises as one runs along it). Where there is life, there is change; where there is change, there is calculus. Differential calculus is specifically concerned with rates of change on an *infinitesimal* scale. Thus, it is not concerned with how the temperature is changing over a period of weeks, years, or centuries, but rather with how the temperature is changing at a given *instant*.

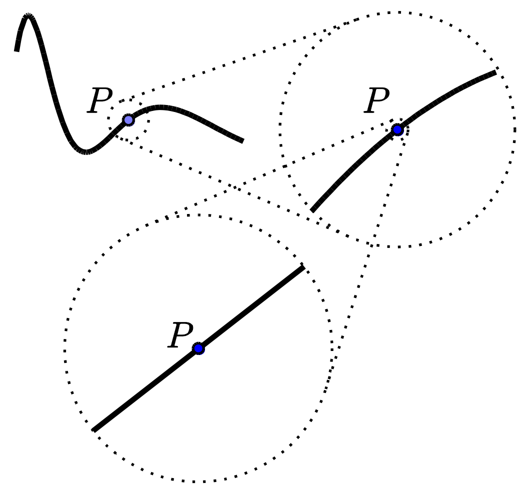
Naturally, the two branches of calculus work together: To understand large-scale global change, we mentally disintegrate it into an infinite sequence of local instantaneous changes; we scrutinize these infinitesimal changes with differential calculus, and then we re-*integrate* them with integral calculus, so as to see the whole again with new eyes and new insights.

In this book, we'll begin with differential calculus, and then move on to integral calculus.

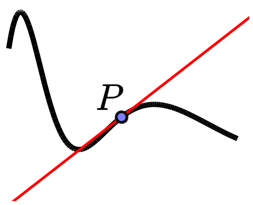
Differential Calculus: The Key Geometric Idea

Differential calculus grows from a single idea: **On an infinitesimal scale, curves are straight**.

To see this, imagine zooming in on a point P lying on a curve. As you do so, the part of the curve you can see (an ever-shrinking "neighborhood" of P) becomes less and less curvy. In an *infinitesimally* small neighborhood of P, the curve coincides with (part of) a straight line.

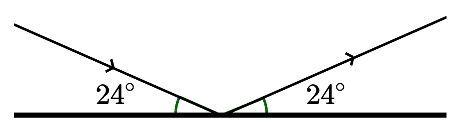


We call this straight line the curve's **tangent at P**. Thus, in the infinitesimal neighborhood of a point, a curve and its tangent are indistinguishable. Outside of this neighborhood, of course, the curve and its tangent usually go their separate ways, as happens in the figure at left. Nonetheless, simply by recognizing that curves possess "local linearity", we can answer seemingly tricky questions in geometry and physics. Consider the following examples, in which tangent lines make unexpected appearances. No calculations are involved, just *thinking* in terms of infinitesimals.

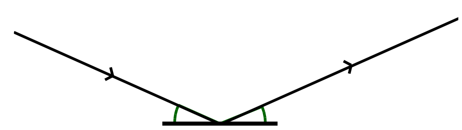


Example 1. (An Illuminating Tangent on Optics)

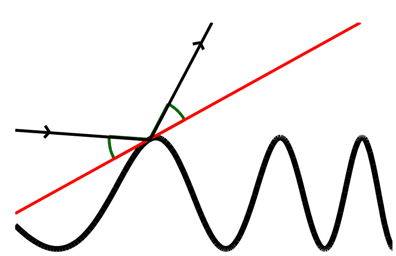
Light reflects off flat surfaces in a very simple manner: Each light ray "bounces off" the surface at the same angle at which it struck it. This much has been known since at least Euclid's time (c. 300 BC). What happens, however, if the surface is *curved*? We shall reason our way to the answer.



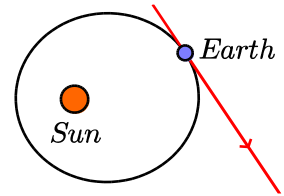
First, note that in the case of a flat surface, we only need an infinitesimal bit of surface against which to measure the relevant angles. (Erasing most of the surface, as in the figure, clearly leaves the angles unchanged.) As far as the light ray is concerned, most of the surface is redundant. The ray's angle of reflection is a strictly local affair, determined entirely in an infinitesimal neighborhood.



So what happens when a light ray strikes a *curve*? Think locally! Let us mentally visit an infinitesimal neighborhood of the light ray's point of impact. There, the curve coincides with its (straight) tangent line, and our curvy conundrum disappears: At this scale, the curve *is* straight, so the old rule still holds! Newly enlightened, we zoom back out to our usual perspective, extend the tangent line as in the figure at right, and know that *this* is the line against which we should measure our reflection angles. ♦



Example 2. (So Long and Thanks for All the Fish.)



If the Sun were to vanish, where would the Earth go? Isaac Newton taught us that the Earth is kept in its orbit by the Sun's gravitational force. He also taught us (following Galileo) that when no forces act on an object moving in a straight line, the object will continue moving along that line. But what about an object moving on a *curved* path that is suddenly freed from all forces that had been acting on it? Well, if we think infinitesimally, we recognize that during any given instant – any infinitesimal interval of time – an object moving on a curved path is actually moving on a *straight* path. Thus when the Sun vanishes, Earth will continue to move along the straight path on which it was traveling in that very instant. That is, the Earth will move along the *tangent* to its elliptical orbit. ♦

Tangents, in short, are important. Let us pause for a few exercises.

Exercises.

1. Let P be a point on a straight line. Describe the tangent to the line at P.

2. Let P be a point on a circle. Describe the tangent to the circle at P. [Hint: Consider the diameter with P as an endpoint. The circle is symmetric about this diameter, so the tangent line at P must also be symmetric about it. (Any "lopsidedness" of the tangent would indicate an asymmetry in the circle – which, of course, doesn't exist.)]

3. a) When two curves (not straight lines) cross, how can we measure the *angle* at which they cross? Explain why your answer is reasonable.

b) How large is the curved angle between a circle and a tangent to the circle? (cf. Euclid, *Elements* 3.16.)

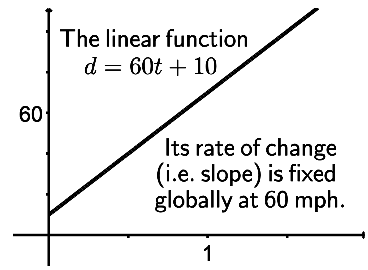
4. If you stand in the open country in eastern Washington, the earth looks like a flat *plane* (hence "the plains"). Of course, it isn't flat; you are actually standing on a sphere. Explain why the earth looks flat from that perspective, and what this has to do with the key idea of differential calculus.

5. Some graphs lack tangents at certain points. Explain why the graph of $y = |x|$ lacks a tangent at its vertex. [*Hint: Reread the first few paragraphs of this section*.] The gleaming calculus machine does what it was designed for phenomenally well, but it was not built to handle corners. At corner points (which, fortunately, are rare on the graphs of the most commonly encountered functions), differential calculus breaks down.

Rates of Change

Once we recognize the local straightness of curves, it affects even the way we think about *functions*. On an infinitesimal scale, any function's *graph* is a straight line, which, in turn, is the graph of a *linear* function. Hence, **on an infinitesimal scale, all functions are linear!***

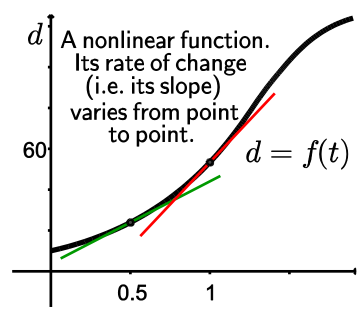
This is excellent news, for linear functions are baby simple. They are simple because the output of any linear function *changes at a fixed rate*, which we call its *slope.* ("Slope" and "rate of change" are more or less synonymous.) If, for example, your car is $60t + 10$ miles from your house t hours after we start a stopwatch, then its distance from your house is changing at the fixed rate of 60 miles per hour. English has a word for the rate at which distance changes in time, so we might as well use it: Your car's *speed* is fixed at 60 miles per hour.



Few phenomena we wish to study are so obliging as to conform to a strict linear relationship. No one actually drives in accordance with the linear function in the previous paragraph – at least not for any appreciable length of time. Rather, a driver's speed varies from moment to moment, even if, on average, he drives 60 miles per hour. But even though physical phenomena are rarely linear in a *global* sense, the functions with which we model them are still *locally* linear. This is precisely why linear functions are so important: Linear functions underlie all functions.

Accordingly, it makes sense to speak of a nonlinear function's *local* rate of change or slope. This notion of a *local* (also called *instantaneous*) rate of change is familiar to every driver; when you glance down at your speedometer, its reading of "63 mph" indicates that your car is moving down the highway at that particular rate *at that particular instant*. Even a few seconds later, your speed may differ. When a policeman aims his speed gun at your car, he is measuring your *instantaneous* rate of change.

To sum up: Locally, the curved graph of a nonlinear function coincides with a straight tangent line. **The tangent's slope is the function's *local* slope, which is the function's rate of change near the point of tangency.** With this insight, we may visually estimate a nonlinear function's local rate of change. For example, consider the figure at right, which is a more realistic *nonlinear* graph of a car's distance travelled (in miles) as a function of time (in hours). The slopes of the two tangents shown in the figure measure the distance function's local rate of change (i.e. the car's speed) at two different times. The graph thus shows us quite plainly that the car was moving faster after one hour than it was after half an hour.



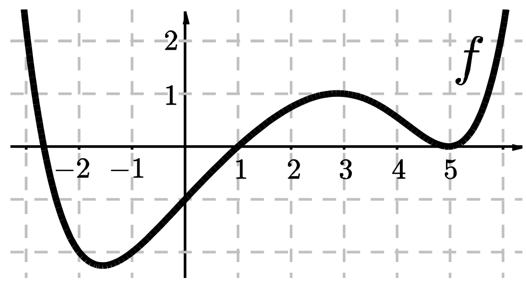
* Exceptions to this rule occur in infinitesimal neighborhoods of "corner points" of the sort described in exercise 5. For the sake of readable (if slightly inexact) exposition, I shall continue to make universal statements about "all functions", trusting the intelligent reader, who has been forewarned, always to bear in mind that calculus breaks down where no tangent line exists.

The Derivative of a Function

We are finally ready to define the central object of differential calculus, the **derivative** of a function.

> **Definition.** The **derivative** of a function is a new function whose output is the original function's *local slope* (or equivalently, *local rate of change*) at the given input value.
>
> If f is a function, its derivative is denoted $\boldsymbol{f'}$ (which we read "f-prime").
> (There are other notations for a function's derivative, which you'll meet in due time.)

This idea is best understood through a few examples.



Example 1. If a function f is given by the graph at right, then according to the derivative's definition, $f'(0)$ measures the graph's *slope* (i.e. the slope of its tangent) when $x = 0$. Clearly, in an infinitesimal neighborhood of $(0, -1)$, this graph resembles a line whose slope is approximately 1, so

$$f'(0) \approx 1.$$

Similarly, we see that $f'(5) \approx 0$, and $f'(4) < 0$. ♦

Example 2. Suppose that after driving for t hours, Jehu's car has travelled a total of $f(t)$ miles. By the derivative's definition, $f'(t)$ represents the *rate* at which his distance is changing at time t. That is, $f'(t)$ represents Jehu's *speed* (in miles per hour) at time t.

If, for example, $f'(1/2) = 112$, then we know that exactly 30 minutes after he began driving, Jehu was driving furiously (112 mph). If, three hours later, traffic brought his car to a standstill, then we have that $f'(7/2) = 0$.

A graph of Jehu's distance function f would have a slope of 112 when $t = 1/2$, and a horizontal tangent line when $t = 7/2$. ♦

Most people use "speed" and "velocity" synonymously, but these have distinct meanings in physics, where "speed" is just a magnitude (a positive number), while "velocity" is a magnitude *with a direction*. When we analyze motion in one dimension, we use positive and negative numbers to specify direction. For example, when we consider just the vertical motion of a ball (disregarding its horizontal motion), a velocity of − 5 ft/sec signifies that the ball is *descending* at 5 ft/sec.

In such contexts, positive and negative numbers also extend the notion of ***distance*** *from a point* (e.g. "3 meters away") to the richer concept of ***position*** *relative to a point* (e.g. "3 meters *to the right*"). For example, if a ladybug paces back and forth on a line, one point of which we call the origin and one direction of which we deem positive, we might have occasion to describe her *position* as $+8$ inches at one moment, and -8 at another, depending on which side of the origin she happens to be. (In both cases, her *distance* from the origin is 8 inches.)

With these distinctions in mind, we can say that the rate of change of *position* with respect to time is *velocity*. Consequently, the *derivative* of an object's position function describes the object's velocity.

Example 3. When Eris tosses a golden apple in the air, its height after t seconds is $h(t)$ meters. Since h is the apple's *position* function (in the up/down dimension), its derivative, $h'(t)$, gives the apple's *velocity* (in m/s) after t seconds.

For example, if we are told that $h(1) = 12$ and $h'(1) = 5$, we may interpret this as follows: After one second, the apple is 12 meters high and rising at 5 m/s. If we also learn that $h(2) = 12$ and $h'(2) = -5$, we know that *two* seconds after leaving Eris's hand, the apple is once again 12 meters high, but it is now on its way down, for it is *descending* at 5 m/s.

When the apple reaches its zenith, its velocity must be *zero*: At that instant, it is neither ascending nor descending, but *hanging*.

Thus, if the apple's maximum height occurs when $t = 3/2$, we'd have $h'(3/2) = 0$. ♦

Velocity is the rate at which position changes. *Acceleration* is the rate at which *velocity* changes. Hence, we measure acceleration in units of velocity per unit time, such $(m/s)/s$ (abbreviated, alas, as m/s^2).*
Note the chain of three functions linked by derivatives: The derivative of an object's *position* function is its *velocity*; the derivative of its velocity function is its *acceleration*.

Acceleration is thus the *second derivative* of position, where "second derivative" simply means "the derivative of the derivative". The second derivative of a function $f(x)$ is denoted, unsurprisingly, $f''(x)$.

Example 4. If we neglect air resistance, any body in free fall (i.e. with no force but gravity acting on it) near Earth's surface accelerates downwards at a constant rate of 32 ft/s^2. Consequently, if $f(t)$ describes the height (in feet) of such an object, and t is measured in seconds, we immediately know that the second derivative of f is a constant function: $f''(t) = -32$. ♦

Position and velocity are rare – but vitally important – examples of functions whose derivatives have special names. Even when such names do not exist, you can always interpret any given derivative as the rate at which its output changes with respect to its input.

Example 5. Suppose $V(t)$ represents the volume (in m^3) of beer in a large vat, where t is the number of hours past noon on a certain day. Throughout the day, beer leaves the vat (as people drink it), but new beer is also poured in by Ninkasi, the Sumerian beer goddess.

Here, $V'(t)$ represents the *rate at which the volume of beer in the vat is changing at time t*. If, say, $V'(5.5) = -0.5$, then at $5{:}30$ pm, the vat's volume is decreasing at a rate of half a cubic meter per minute. (Even Ninkasi struggles to satiate thirsty Sumerians after the 5 o'clock whistle.) Horrors! Will the beer run out? Well, suppose we also learn that $V(6) = 0.0001$ and $V'(6) = 2$. These values imply that at $6{:}00$, the vat is dangerously close to empty, but – at that very instant – it is also filling back up at a torrential rate of 2 cubic meters per minute. All praise to Ninkasi!

Whatever the graph of $V(t)$ may look like overall, we know it must pass through $(6, 0.0001)$, dropping almost down to the t-axis (which would signify an empty vat), but at that very point, we know the graph must also exhibit a strong sign of recovery: an upward-thrusting slope of 2. ♦

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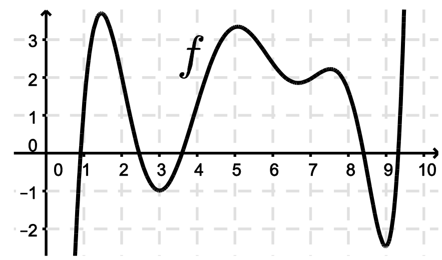
* *Squared seconds* are not physically meaningful units, hence the parenthetical sigh. One must, like Leopold Bloom, remember what m/s^2 actually *means*: "Thirtytwo feet per second, per second. Law of falling bodies: per second, per second. They all fall to the ground… Per second, per second. Per second for every second it means." (James Joyce, *Ulysses*)

Exercises.



6. Judging by the graph at right…

a) What is the approximate numerical value of $f'(3)$?

b) Arrange in numerical order: $f'(1),\ f'(2),\ f'(3),\ f'(4),\ f'(6)$.

c) How many solutions does $f(x) = 0$ have in the interval $[2,8]$?

d) How many solutions does $f'(x) = 0$ have in the interval $[2,8]$?

e) Which *integers* in $[2,8]$ satisfy the inequality $f(x)f'(x) < 0$?

f) True or false: $f'(\pi) > 0$. g) True or false: $[f(5)]/3 > f'(7)$.

7. True or false: If $f(x) = x^2$, $g(x) = \ln x$, and $h(x) = 1/x$, then…

a) $f'(0) = f(0)$. b) $g(x) > 0$ for all x in the domain of g. c) $g'(x) > 0$ for all x in the domain of g.

d) $g(1) = g'(1)$. e) $h'(x) < 0$ for all x in the domain of h. f) $h'(-1) = h'(1)$.

g) $f'(x) > g'(x)$ for all positive values of x.

8. Given the functions in the previous problem, arrange the following in numerical order: $f'(5),\ g'(5),\ h'(5)$.

9. Consider the constant function $f(x) = 5$. What is $f'(0)$? What is $f'(5)$? What is $f'(x)$? What can be said about the derivative of a constant function in general?

10. Consider the general linear function $g(x) = ax + b$. What is $g'(x)$?

11. The function $h(x) = |x|$ is defined for all real values of x, but $h'(x)$ has a slightly smaller domain. What is it? [*Hint*: *See exercise* 5.] Also, thinking geometrically, write down a formula for $h'(x)$.

12. Let $f(x) = \sqrt{4 - x^2}$.

a) Sketch graphs of f and f' on the same set of axes.

[*Hint: If you don't know the graph of f, square both sides of $y = \sqrt{4 - x^2}$; you'll know* ***that*** *equation's graph. Then observe that for each x-value on this "squared" graph, there are two y-values: one positive, one negative. Hence, if we solve its equation for its positive y-values (to recover the strictly positive function $y = \sqrt{4 - x^2}$), half of the graph will disappear; what remains is the graph of f. With that in hand, you can sketch the graph of f' by staring at the graph of f and thinking broadly about how its slope changes as x varies in its domain*.]

Find the exact values of the following.

b) $f'(0)$ c) $f'(1)$ d) $f'(\sqrt{2})$ e) $f'(\sqrt{3})$

[*Hint: Think about exercise* 2*, and recall how perpendicular lines' slopes are related.*]

13. Galileo fires a physics textbook out of a cannon. After t seconds, its height will be $h = -4.9t^2 + v_0 t$, where v_0 represents the book's initial upwards velocity in meters per second. Obviously, the book attains each height in its range (apart from its maximum height) at two separate moments: once going up, once coming down.

Remarkably, the book's speed will be the same at both moments. Without solving equations, explain why. [*Hint: Think geometrically. What does the height function's graph look like? How is speed encoded in it?*]

14. Rube Waddell throws a baseball at the full moon. Let $p(t)$ be the ball's height (in feet) t seconds after it leaves his hand. In terms of physics…

a) What does the quantity $p'(2)$ represent? b) What does the solution to the equation $p'(t) = 0$ represent?

c) What is the formula for $p''(t)$, and why is this so? d) If $p'(a) < 0$, then what is happening at time a?

15. Buffalo Bill goes ice skating. More particularly, he skates along a narrow frozen river running East/West, often reversing his direction by executing beautifully precise $180°$ turns. If we take a cigar that he dropped on the ice to be the origin, and we let East be the positive direction, then his position after t minutes can be described by the function $f(t)$, where distances are measured in meters. In physical terms…

a) What does $f'(t)$ represent? b) What does $|f'(t)|$ represent? c) What does $|f(t)|$ represent?
d) Could there be a time a at which $f(a) > 0$, but $f'(a) < 0$? Explain.
e) If $f''(t) = 0$ for all t in some interval (b, c), what is happening between $t = b$ and $t = c$?
f) If $f'(t) = 0$ for all t in some interval (d, e), what is happening between $t = d$ and $t = e$?
g) Suppose that $f'(t) < 0$ for all t in some interval (m, n), that $f'(n) = 0$, and that $f'(t) > 0$ for all t in some interval (n, p). What happened when $t = n$?

16. Physicists call the rate of change of acceleration *the jerk*. (Thus, the jerk is position's *third* derivative.)
a) If distance is measured in meters, and time in seconds, what are the units of the jerk?
b) What, if anything, can be said about a freely-falling object's jerk?

17. Suppose that the function $T(h)$ gives the temperature on 11/1/2015 at 10:46 am (the time at which I'm typing these words) h meters above my house in Olympia, WA. Suppose further that the function $T'(h)$ is strictly negative (meaning that its value is negative for all heights h). What does this strictly negative derivative signify physically?

18. Buzz Aldrin is walking clockwise around the rim of a perfectly circular crater on the moon, whose radius is 2 miles. Let $s(t)$ be the distance (in miles, as measured along the crater's rim) he has walked after t minutes.

a) If $s'(t) = \pi/40$ for all t during his first lap, then how long will it take him to complete one lap?
b) If $s'(t) > 0$ and $s''(t) < 0$ throughout his second lap, will Buzz be walking faster when he begins his second lap or when he ends it?

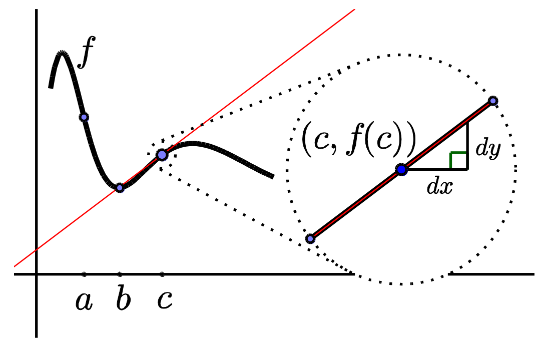
19. A mysterious blob from outer space has volume $V(t)$ after t hours on Earth, where V is measured in cubic feet.
a) What does $V'(5)$ represent physically, and in what units is it measured?
b) If the graph of $V(t)$ shows that V is decreasing between $t = 24$ and $t = 48$, what do we know about $V'(t)$?

A Gift From Leibniz: $\boldsymbol{d}$-Notation

Now that you know what a derivative is, you are ready for the special notation introduced by Gottfried Wilhelm Leibniz, one of the intellectual giants in the story of calculus.

You are no doubt familiar with the "delta notation" commonly used to describe the *change* in a variable.* Recall in particular that we find a line's slope as follows: From the coordinates of any two of its points, we compute the "rise" (Δy) and "run" (Δx); the slope is then "the rise over the run", $\Delta y/\Delta x$.

In calculus, we use Leibniz's analogous "d-notation" to describe *infinitesimal* change. If m represents a magnitude (length, temperature, or what have you), then the symbol **$\boldsymbol{dm}$ represents an *infinitesimal change* in $\boldsymbol{m}$.**



Since a curve is straight on an infinitesimal scale, the slope of its tangent (i.e. its derivative) at any given point is given by the ratio of infinitesimals $\boldsymbol{dy/dx}$, as in the figure. Accordingly, if $y = f(x)$, we frequently use "dy/dx" as an alternate notation for the derivative; that is, in place of $f'(x)$, we often write dy/dx.

The two derivative notations peacefully coexist. We use whichever one is more convenient in a given situation.

Prime notation's prime advantage is that it provides a clear syntactic place for the derivative's input. For example, consider the figure above. When the input is c, the derivative of f is approximately $3/4$. Using prime notation, we can express this fact quite compactly: $f'(c) \approx 3/4$. In contrast, expressing this same fact in **Leibniz notation** (as his d-notation is now called) is a bit more bothersome, as we must also supply a brief explanatory phrase:

$$\frac{dy}{dx} \approx 3/4 \text{ when } x = c.$$

The concluding phrase is essential, since dy/dx's values vary.†

Leibniz notation's many advantages will become ever clearer as you learn more and more calculus. For now, you should just appreciate how dy/dx encapsulates the derivative's *meaning*. When we see a derivative represented symbolically as dy/dx (an infinitesimal rise over an infinitesimal run), we are reminded that a derivative is a slope, a rate of change. In contrast, prime notation is totally arbitrary. After all, why use a prime and not, say, a dot?‡

Leibniz understood the importance of mental ergonomics. When he designed notation, he tried to fit it to the mind's natural contours. As we proceed through the course, we'll encounter numerous instances in which his notation will seem to do half of our thinking for us. Be thankful for Leibniz's gift.

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* Examples: If h, the height of a sunflower (in feet), changes from 3 to 7 over a period of time, then $\Delta h = 4$ feet.
If the temperature T drops from 9° to $-11°$, then $\Delta T = -20°$.

† The notation $\left.\frac{dy}{dx}\right|_c$ is occasionally used for the derivative's value at c, but is very awkward. I won't use it in this book.

‡ Newton, in fact, did use a dot for derivatives. Newton and Leibniz are usually credited as the independent co-creators of calculus, which oversimplifies history a great deal, but is still a good first approximation to the truth. A bitter priority dispute arose between these two men and their followers.

Exercises.

20. If $y = f(x)$, rewrite the following statements in Leibniz notation: $f'(2) = 3$, $f'(\pi) = e$.

21. If $y = f(x)$, rewrite these statements in prime notation: $\frac{dy}{dx} = 1$ when $x = -2$, $\frac{dy}{dx} = \ln 5$ when $x = \sqrt{5}$.

22. If $f(2) = 6$ and $f'(2) = 3$ for a function f, then in an infinitesimal neighborhood around $x = 2$, the function's output increases by 3 units for each 1 unit of increase in the input. (Be sure you see why: Think geometrically.)

For most functions, the preceding statement is still approximately true for small *real* (i.e. not infinitesimal) neighborhoods of $x = 2$. Thus, increasing the input variable from 2 to 2.001 should cause the output variable to increase from 6 to approximately $6 + 3(.001) = 6.003$.

a) Convince yourself that the approximation described in the preceding paragraph should indeed be reasonable (even though the neighborhood isn't infinitesimal), provided the graph of f isn't too intensely curved near the point $(2, 6)$. As always, draw pictures to aid your intuition.

b) If $f(5) = 8$, and $f'(5) = 2.2$, approximate the value of $f(5.002)$.

c) If $g(0) = 2$, and $g'(0) = -1.5$, approximate the value of $g(0.003)$.

d) If $s(x) = \sin x$ (where x is measured in radians), we'll soon be able to prove that $s'(\pi) = -1$. Assuming this fact for now, approximate the value of $\sin(3.19)$. Then check your answer with a calculator.

e) TRUE of FALSE (explain your answer): If h is a function such that $h(3) = 2$ and $h'(3) = 4$, it is reasonable for us to assume that $h(10) \approx 2 + 4(7) = 30$.

23. We often read dy/dx aloud as "the derivative of y *with respect to* x". Similarly, dz/dq is the derivative of z with respect to q. In applications, this feature of Leibniz notation helps us keep track of units of measurement. For example, if K represents an object's kinetic energy (in joules) at time t (in seconds), then dK/dt is the derivative of kinetic energy *with respect to time*; it tells us how kinetic energy changes *in response to changes in time*. Moreover, the Leibniz notation makes it clear that dK/dt is measured in joules per second.

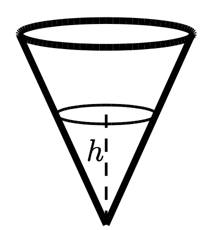
a) If, in the preceding kinetic energy example, we know that when $t = 60$, we have $K = 80$ and $dK/dt = 12$, then roughly how much kinetic energy might we reasonably expect the object to have when $t = 60.5$?

b) Suppose that A represents an area that grows and shrinks over time. Use Leibniz notation to express the following: After 5 minutes, the area is shrinking at a rate of 2 square meters per minute.

c) If $z = g(t)$, rewrite $g'(6) = 4$ in Leibniz notation.

d) Let C be the total cost (in dollars) of producing Q widgets per year. Boosting Q past certain values might require costly changes such as purchasing more machinery or hiring more employees. The derivative dC/dQ is known in economics as the "*marginal* cost function". In what units would values of dC/dQ be measured?

24. Leibniz notation's explicit reference to the derivative's input variable (see the previous exercise) is especially useful when "the" input variable can be viewed in multiple ways.



Consider, for instance, a conical vat. Suppose water pours into the (initially empty) vat at a constant rate of 3 ft^3/min. Let V be the volume of water in the cone, and let h be the water's "height", as indicated in the figure. Naturally, we can view V as a function of t, the time elapsed since the water began pouring in. However, we can also view V as a function of h; if the water's height is known, then the volume of the water is, in principle, determined – regardless of whether you know *how* to determine its numerical value.

Since V can be considered a function of t or h, we can distinguish between two different derivatives of V: dV/dt and dV/dh. The former measures the rate at which the volume changes with respect to *time*. This, we are told, is constant: $dV/dt = 3$ ft^3/min. The latter, dV/dh, measures the rate at which the volume changes with respect to the water's *height*. A little thought will convince you that this is *not* a constant rate of change.

a) Have the little thought mentioned in the previous sentence. Namely, to convince yourself that dV/dh is *not* a constant function of h, imagine two different situations corresponding to different values of h. First, let h be very small, so that there is hardly any water in the vat. If, in this case, we increase h by a tiny amount dh, consider the resulting change in volume, dV. *Draw a picture, and indicate what* dV *represents geometrically*. (You need not calculate anything.) Second, let h be relatively large, so that the vat is, say, $3/4$ full. Again, imagine increasing h by the same tiny amount dh. *Draw another picture and think about what* dV *represents geometrically*. Since applying the same little nudge to the input variable dh yields different changes to the output variable, dV, the ratio dV/dh has a different value in each case. Hence, dV/dh is not constant.

b) Which is greater: dV/dh when h is small, or dV/dh when h is large?

c) If the cone were upside down, so that water poured into its vertex at a constant rate, which would be greater: dV/dh when h is small, or dV/dh when h is large? *Draw a picture.*

d) If the water were pouring into a *spherical* tank of radius 10 feet, rank the following in numerical order: dV/dh when $h = 1$, dV/dh when $h = 5$, dV/dh when $h = 10$, dV/dh when $h = 17$.

e) Give an example of a shape for a tank that would ensure that dV/dh is constant if dV/dt is constant.

25. When $y = f(x)$, we can write dy/dx in the form $d(f(x))/dx$. Thus, for example, we may rewrite

$$\text{If } y = \tan x, \text{ then } \frac{dy}{dx} = 4 \text{ when } x = \frac{\pi}{3},$$

in the following more concise form:

$$\frac{d(\tan x)}{dx} = 4 \text{ when } x = \frac{\pi}{3}.$$

Naturally, all the usual interpretations hold. Here, for example, the notation is telling us that in the infinitesimal neighborhood of $x = \pi/3$, the output value of $\tan x$ increases by four units for each unit by which its input x is increased. Use this notation to rewrite the following statements.

a) If $y = \ln x$, then $\frac{dy}{dx} = 5$ when $x = \frac{1}{5}$.

b) If $y = 3x^3 + 1$, then $\frac{dy}{dx} = 36$ when $x = 2$.

c) If $y = 2^x$, then $\frac{dy}{dx} = \ln 16$ when $x = 2$.

d) If $y = -4x + 2$, then $\frac{dy}{dx} = -4$.

e) In part d, why wasn't it necessary to include a qualifying statement about an x-value? *Think geometrically*.

An Infinitesimal Bit of an Infinitesimal Bit

"So naturalists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller fleas to bite 'em.
And so proceed *ad infinitum*."
- Jonathan Swift, "On Poetry: A Rhapsody"

Never forget: Ultimately, we are interested in *real*-scale phenomena. We work with infinitesimal bits of real magnitudes (dx, dz, or what have you) precisely because they help us understand real phenomena. In contrast, infinitesimal bits *of infinitesimals* ("second-order infinitesimals" such as $(dx)^2$ or $du \cdot dv$) mean nothing to us; when they appear in the same context as real magnitudes, we simply disregard them as though they were zeros. For instance, if we expand the binomial $(x + dx)^2$ to obtain

$$(x + dx)^2 = x^2 + 2x(dx) + (dx)^2,$$

we treat the second-order infinitesimal $(dx)^2$ as a zero, and thus we write $(x + dx)^2 = x^2 + 2x(dx)$.

It helps to imagine that the "calculus microscope" we described earlier (on the chapter's first page) can magnify first-order infinitesimals to visibility, but is too weak to detect higher-order infinitesimals. This "weakness" actually puts our eye in an ideal position – neither too close, nor too far away from the real magnitudes that we wish to describe. Could a more powerful microscope make sense of higher-order infinitesimals? Perhaps, but we need not concern ourselves with such questions here; we are interested in infinitesimals not for their own sake, but rather for what they tell us about ordinary, real-scale phenomena. For that purpose, first-order infinitesimals suffice.

Exercises.

26. Expand the following.

a) $(x - dx)^2$ b) $(x + dx)^2 - x^2$ c) $(x + dx)^3$ d) $(u + du)(v + dv)$

27. If we increase a function f's input from x to $x + dx$, its output changes from $f(x)$ to $f(x + dx)$. Consequently, the expression $\boldsymbol{d(f(x))}$, which denotes the infinitesimal change in f's value, is $\boldsymbol{f(x + dx) - f(x)}$.

[Thus, for example, $d(x^2) = (x + dx)^2 - x^2$. And so, by exercise $26b$, $d(x^2) = 2xdx$.]

Your problem: Show that…

a) $d(x^3) = 3x^2dx$ b) $d(3x^2) = 6xdx$ c) $d(ax^2 + bx + c) = (2ax + b)dx$.

28. Expand the following infinitesimal changes by writing them as *differences*.

a) $d(f(x))$ b) $d(5f(x))$ c) $d(f(x) + g(x))$ d) $d(f(x)g(x))$ e) $d\left(f(g(x))\right)$

29. On the graph of $y = x^2$, Let P be the point (3,9). Let Q be a point, infinitesimally close to P, whose coordinates are $(3 + dx,\ (3 + dx)^2)$. As we move from P to Q, the infinitesimal change in x is dx.

a) Express dy, the corresponding infinitesimal change in y, in terms of dx.

b) Since $y = x^2$, we have $dy = d(x^2)$. Use this, and your result from part (a) to compute $\frac{d(x^2)}{dx}$ when $x = 3$.

c) Use the ideas in this problem to find $\frac{d(x^2)}{dx}$ when $x = -1/2$.

30. a) Show that $\dfrac{\frac{u+du}{v+dv} - \frac{u}{v}}{dx} = \dfrac{vdu - udv}{v^2dx}$ b) Show that the right-hand side can be rewritten as $\dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$.

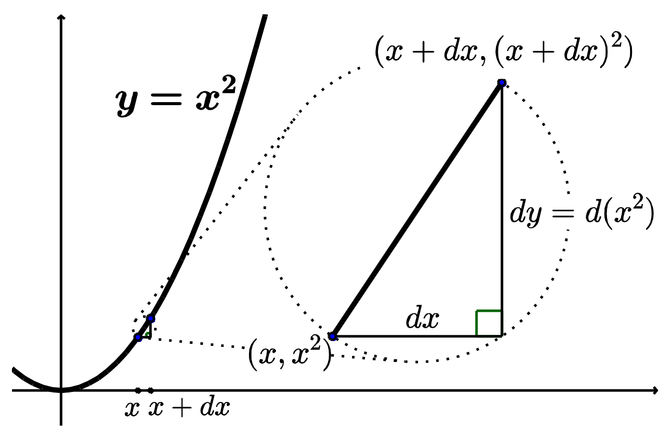
The Derivative of $y = x^2$

Differential calculus teaches us nothing about linear functions; if we put the graph of a linear function under the calculus microscope, we see the same old straight line that had been visible to the naked eye. In exercise 10, you learned all there is to know about linear functions' derivatives: Linear functions have constant slopes, so their derivatives are constant functions. End of story.

Calculus exists to analyze *nonlinear* functions. Perhaps the simplest nonlinear function is $y = x^2$, whose derivative we'll now compute. This will be our first nontrivial example of a derivative. Note well: We are not merely looking for this function's derivative at a particular point (say, dy/dx when $x = 3$), which is a *number* (such as you found in exercise 29b); we seek the derivative dy/dx itself, a *function*. Once we've found a formula for dy/dx, we can evaluate it wherever we wish.

Problem. Find the derivative of the function $y = x^2$.

Solution. At each point x in the function's domain, the derivative's output will be the function's local rate of change there, dy/dx. To compute this ratio, we observe that when x is increased by an *infinitesimal* amount dx (which must be exaggerated in the figure!), the corresponding change in y is



$$\begin{aligned} dy &= (x + dx)^2 - x^2 \\ &= 2x(dx).^* \end{aligned}$$

Thus, for any x in the domain, we have

$$\frac{dy}{dx} = \frac{2x(dx)}{dx} = \boldsymbol{2x}. \quad ♦$$

We've just proved that the derivative of x^2 is $2x$. That is,

$$\frac{d(x^2)}{dx} = 2x.$$

We often state results of this type ("the derivative of A is B") in the following alternate form:

$$\frac{d}{dx}(x^2) = 2x.$$

Here, we think of the symbol $\boldsymbol{d/dx}$ as an *operator* that turns a function into its derivative.†

Our first substantial project in differential calculus, which we'll begin after the next set of exercises, will be to discover a method for rapidly finding the derivative of any polynomial whatsoever.

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* See exercise $26b$ above.

† Just as a function turns numbers into numbers, an *operator* turns functions into functions.

Exercises.

31. Knowing $\frac{d(x^2)}{dx} = 2x$ in general, you should need only seconds to find $\frac{d(x^2)}{dx}$ when $x = 4$ in particular. Find it.

32. Find the equation of the line tangent to the graph of $y = x^2$ at point $(-3,9)$.

33. a) Find the coordinates of the one point on the graph of $y = x^2$ at which the function's slope is exactly -5.
b) Find the equation of the tangent to the graph at the point you found in part (a).

34. Is there a non-horizontal tangent to $y = x^2$ that passes through $(1/3, 0)$? If so, find the point of tangency.

35. Use the ideas in this section to find $\frac{d(ax^2 + bx + c)}{dx}$.

36. a) Use the ideas in this section to find $\frac{d(x^3)}{dx}$. b) Use part (a) to find $\frac{d(x^3)}{dx}$ when $x = \sqrt{2}$, and when $x = \pi$.
c) Find the equation of the line tangent to the graph of $y = x^3$ at point $(1,1)$.
d) Does the tangent line from part (c) cross the graph again? If so, where? If not, how do you know?
[*Hint: At some point, you'll need to solve a cubic equation. Even though you (presumably) do not know how to solve cubics in general, you can solve this particular one because you already know one of its solutions*.]

37. The notations $\frac{d(f(x))}{dx}$ and $\frac{d}{dx}(f(x))$ are equivalent and are used interchangeably. To accustom yourself to these notational dialects, rewrite the following expressions in the other form. (Yes, this exercise is trivial.)

a) $\frac{d(x^2)}{dx}$ b) $\frac{d(\sin x)}{dx}$ c) $\frac{d}{dx}(\ln x)$ d) $\frac{d}{dx}\left(\frac{1}{x}\right)$

Derivatives of Polynomials

'He is like a mere x. I do not mean x the kiss symbol but, as we allude in algebra terminology, to denote an unknown quantity.'
'What the hell this algebra's got to do with me, old feller?'
- *All About H. Hatterr*, G.V. Desani.

Let us revisit some basic algebra you learned on your mother's knee.

How does one multiply algebraic expressions such as $(a + b + c)(d + e + f)$? Well, each term in the first set of parentheses must "shake hands" with each term in the second set. The *sum* of all such "handshakes" (i.e. multiplications) is the product we seek.

For example, to multiply $(a + b + c)(d + e + f)$, first a shakes hands with each term in the second set (yielding ad, ae, and af), then b does (bd, be, and bf), and finally, c does (cd, ce, and cf). Thus,

$$(a + b + c)(d + e + f) = ad + ae + af + bd + be + bf + cd + ce + cf.$$

Naturally, this works regardless of how many terms are in each parenthetical expression. Applied, for instance, to the very simple product $(a + b)(c + d)$, the handshake game produces the familiar "FOIL" expansion you learned in your first algebra course.

If one wishes to multiply not two, but *three* expressions, then each "handshake" must be a *three-way* handshake, with one handshaker drawn from each of the three expressions. For example,

$$(a + b)(c + d)(e + f) = ace + acf + ade + adf + bce + bcf + bde + bdf.$$

Similarly, multiplying four expressions requires four-way handshakes, while multiplying n expressions, as we'll need to do in a moment, requires n-way handshakes. Let us rest our hands and return to calculus.

Problem. Find the derivative of the function $y = x^n$, where n is a whole number.

Solution. We begin by noting that

$$\frac{d(x^n)}{dx} = \frac{(x+dx)^n - x^n}{dx}.$$

To proceed, observe that the binomial in the numerator is

$$(x+dx)^n = \overbrace{(x+dx)(x+dx)\cdots(x+dx)}^{n \text{ times}}.$$

To multiply this out, we must sum all possible n-way "handshakes" of the sort described above.

The simplest of these n-way handshakes will involve all n of the x's and none of the dx's. This handshake's contribution to the expansion of $(x+dx)^n$ is obviously $\boldsymbol{x^n}$.

Among the many other n-way handshakes, some will involve $(n-1)$ of the x's and one dx. In fact, there will be exactly n handshakes of this sort. (The first of them involves the dx from the *first* parenthetical expression and the x's from all the other groups; the second involves the dx from the *second* parenthetical expression and the x's from all the other groups – and so forth.) Since each of these n handshakes will add an $x^{n-1}dx$ to the expansion of $(x+dx)^n$, their net contribution to the expansion will be $\boldsymbol{nx^{n-1}dx}$.

Each of the remaining handshakes involves at least two dx's, so their contributions to the expansion will be higher-order infinitesimals, which means that we can simply disregard them! Therefore, the expansion we seek is $(x+dx)^n = x^n + nx^{n-1}dx$.

Consequently,

$$\begin{aligned}\frac{d(x^n)}{dx} &= \frac{(x+dx)^n - x^n}{dx} \\ &= \frac{(x^n + nx^{n-1}dx) - x^n}{dx} \\ &= nx^{n-1}.\end{aligned}$$

♦

The result we've just established is important enough to merit its own box.

The Power Rule (Preliminary Version).

For all whole numbers n,

$$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}.$$

This is a preliminary version inasmuch as we will eventually prove that the power rule holds not just for whole number powers, but for all *real* powers whatsoever.

Example. By the power rule, $\frac{d}{dx}(x^7) = 7x^6$. ♦

A quick note on language: We do *not* say that we "derive" x^7 to obtain its derivative, $7x^6$. Rather, we "take the derivative of" x^7.* Many students commit this faux pas. Don't be one of them.

Over the next few chapters, you'll learn how to take many functions' derivatives. Along with derivatives of specific functions (x^n, $\sin x$, $\ln x$, etc.), you'll learn *structural* rules that let you take derivatives of nasty functions built up from simple ones (such as $x^2 \ln x + 4\sin(5x)$). The first structural rules we'll need are the derivative's *linearity properties*.

Linearity Properties of the Derivative.

i. The derivative of **a constant times a function** is the constant times the function's derivative:

$$\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}\big(f(x)\big).$$

ii. The derivative of **a sum of functions** is the sum of their derivatives:

$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big).$$

Proof. To prove the first linearity property, we just follow our noses:

$$\begin{aligned}\frac{d}{dx}\big(cf(x)\big) = \frac{d\big(cf(x)\big)}{dx} &= \frac{cf(x+dx)-cf(x)}{dx} \\ &= c\left(\frac{f(x+dx)-f(x)}{dx}\right) = c\left(\frac{d\big(f(x)\big)}{dx}\right) = c\frac{d}{dx}\big(f(x)\big).\end{aligned}$$

Proving the second property is just as simple.

$$\begin{aligned}\frac{d}{dx}\big(f(x)+g(x)\big) &= \frac{d\big(f(x)+g(x)\big)}{dx} = \frac{\big(f(x+dx)+g(x+dx)\big)-\big(f(x)+g(x)\big)}{dx} \\ &= \frac{\big(f(x+dx)-f(x)\big)+\big(g(x+dx)-g(x)\big)}{dx} = \frac{f(x+dx)-f(x)}{dx} + \frac{g(x+dx)-g(x)}{dx} \\ &= \frac{d\big(f(x)\big)}{dx} + \frac{d\big(g(x)\big)}{dx} = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big). \quad \blacksquare\end{aligned}$$

If that proof gave you any trouble, please work through it again after revisiting exercises 27, 28, and 37. The justification for each equals sign in the proof should be crystal clear to you.

Using the power rule and the linearity properties, we can find any polynomial's derivative in a matter of seconds, as the following example and exercises will convince you.

Example. Find the derivative of $y = 5x^6 + 3x^4$.

Solution. $\frac{d}{dx}(5x^6+3x^4) = \frac{d}{dx}(5x^6) + \frac{d}{dx}(3x^4)$ (by one of the linearity properties)

$= 5\frac{d}{dx}(x^6) + 3\frac{d}{dx}(x^4)$ (by the other linearity property)

$= 5(6x^5) + 3(4x^3)$ (by the power rule)

$= 30x^5 + 12x^3$. ♦

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* One may also use the verb *differentiate* here. (E.g. If we differentiate x^7, we obtain $7x^6$.) However, since this sense of "differentiate" has nothing in common with the verb's ordinary meaning, those who use it must take care to differentiate differentiate from differentiate.

Exercises.

38. The power rule is geometrically obvious in the special cases when $n = 1$ or $n = 0$. Explain why.

39. If we combine the power rule and the first linearity property, we find that $\frac{d}{dx}(cx^n)$ is equal to what?

40. Use the result of exercise 39 to find the following derivatives in one step:

a) $\frac{d}{dx}(5x^3)$ b) $\frac{d}{dx}(-3x^7)$ c) $\frac{d}{dx}\left(\frac{\pi}{2}x^2\right)$

41. By combining the result of exercise 39 and the second linearity property, find the following derivatives.

a) $\frac{d}{dx}(3x^3 + 2x^2)$ b) $\frac{d}{dx}\left(-10x^5 + \frac{1}{4}x^3\right)$ c) $\frac{d}{dx}\left(\sqrt{2}x + \sqrt{3}\right)$

42. Prove that the derivative of **a difference of functions** is the difference of their derivatives.

43. Use the result of the previous exercise to find the following derivatives.

a) $\frac{d}{dx}(5x^3 - 2x^4)$ b) $\frac{d}{dx}(3x^2 - 5)$ c) $\frac{d}{dx}\left(-\frac{2}{5}x^{10} - \pi\right)$

44. Convince yourself that the derivative of a sum of *three* (or more) functions is the sum of their derivatives. Then use this fact to compute the following, ideally writing down each derivative in a single step:

a) $\frac{d}{dx}(2x^3 + 4x^2 + 5x + 1)$ b) $\frac{d}{dx}\left(\frac{3}{4}x^3 - 9x^2 - \sqrt{5}x + 2\right)$ c) $\frac{d}{dx}(-x^6 + x^3 - 4x^2 + 3)$

45. The derivative of a product of functions is *not* the product of their derivatives! Show, for example, that

$$\frac{d}{dx}(x^5x^3) \neq \left(\frac{d}{dx}(x^5)\right)\left(\frac{d}{dx}(x^3)\right).$$

46. a) True or false: $\frac{d}{dx}((2x+1)^3) = 3(2x+1)^2$. Explain your answer.

b) What, in fact, is the derivative of $y = (2x+1)^3$?

47. The derivative of a quotient of functions is *not* the quotient of their derivatives! Demonstrate this by providing a counterexample, as in exercise 45.

48. If $y = 2\pi^3$, what is dy/dx?

49. What does the power rule tell us about $\frac{d}{dx}(3^x)$?

50. Using symbols other than x and y for a function's independent and dependent variables does not change the formal rules for finding derivatives. With this in mind, find the derivatives of these functions:

a) $f(t) = 2t^3 - 3t^2 + 5$ b) $g(z) = \frac{1}{4}z^4 + \frac{1}{3}z^3 + \frac{1}{2}z^2 + z$ c) $A(r) = \pi r^2$

51. Recall that the graph of any quadratic function (of the form $y = ax^2 + bx + c$) is a *parabola* whose axis of symmetry is parallel to the y-axis. Clearly, such a graph has a horizontal tangent only at its vertex. This observation yields a quick way to find the vertex's x-coordinate: Set the quadratic's derivative equal to zero. Be sure you understand this idea, then use it – together with the fact that a parabola of this sort opens up or down according to whether its leading coefficient is positive or negative – to sketch graphs of the following quadratics. Include the coordinates of each parabola's vertex and of any intersections with the axes.

a) $y = x^2 + 3x + 4$ b) $f(x) = -2x^2 + 3x - 4$ c) $g(x) = \pi x^2 + ex + \sqrt{2}$

52. Find the equation of the line tangent to $y = x^3 - 2x^2 + 3x + 1$ at $(1,3)$.

53. There is exactly one tangent to $y = x^3$ that passes through $(0,2)$. Find the point of tangency.

One Last Example

"Imagine, if you will, that the stone, while in motion, could think… Such a stone, being conscious merely of its own endeavor… would consider itself completely free, would think it continued in motion solely by its own wish. This then is that human freedom which all men boast of possessing, and which consists solely in this: that men are conscious of their own desire, but ignorant of the causes whereby that desire has been determined."

- Spinoza, in a letter to G.H. Schuller (October, 1674)

Your new ability to take polynomials' derivatives lets you solve otherwise tricky applied problems.

Example. Spinoza stands at a cliff's edge, 100 feet above the ocean, and hurls a stone. Its height (relative to the ocean) after t seconds is given by the formula $s(t) = -16t^2 + 64t + 100$.
Answer the following questions.

a) Find the stone's vertical velocity when $t = 0.5$ and when $t = 2.5$.
b) What is the stone's maximum height?
c) How fast is the stone moving downwards at the instant when it hits the water?

Solution. Since $s(t)$ gives the stone's vertical position (height in feet) after t seconds, its derivative, $s'(t) = -32t + 64$, gives the stone's vertical *velocity* (in ft/sec) after t seconds.

Thus, half a second after leaving Spinoza's hand, the stone's vertical velocity is $s'(0.5) = 48$. That is, at that particular instant, it is moving *upwards* at 48 ft/sec. After 2.5 seconds, its vertical velocity is $s'(2.5) = -16$. Hence, at *that* instant, the stone is moving *downwards* at 16 ft/sec.

Clearly, the stone will rise for a while (have positive vertical velocity), then fall (have negative vertical velocity). The stone will reach its maximum height at the instant when it has stopped rising, but has not yet begun to fall. This occurs when its vertical velocity is *zero*. Solving $s'(t) = 0$, we find that this maximum height occurs at $t = 2$. Consequently, the stone's maximum height will be $s(2) = 164$ feet above the ocean.

The stone hits the water when $s(t) = 0$. This is a quadratic equation; substituting its sole positive solution, $t = 2 + \sqrt{41}/2$ into our velocity function, we find that the stone's vertical velocity upon impact is $s'\left(2 + \sqrt{41}/2\right) \approx -102.4$ ft/sec. ♦

As discussed earlier, velocity's rate of change is *acceleration*. In the preceding example, the stone's position was given by $s = -16t^2 + 64t + 100$, from which we deduced its velocity function:

$$v = \frac{ds}{dt} = -32t + 64\,.$$

By taking the derivative of the velocity function, we can now determine the rock's acceleration function:

$$a = \frac{dv}{dt} = -32\,,$$

which agrees with Galileo's famous discovery: Any object in free fall (i.e. with no force other than gravity acting upon it) accelerates downwards at a constant rate of 32 feet per second per second.

Exercises.

54. Suppose a point is moving along a horizontal line. Define *right* as the positive direction. If the point's position (relative to some fixed origin) after t seconds is given by $s = -5 + 4t - 3t^2$, find the time(s) at which the point is momentarily at rest (i.e. when its velocity is zero), the times when the point is moving to the right, and the times when it is moving to the left.

55. Suppose two points are moving on the line from the previous problem, and their positions are given by

$$s_1(t) = t^2 - 6t \quad \text{and} \quad s_2(t) = -2t^2 + 5t.$$

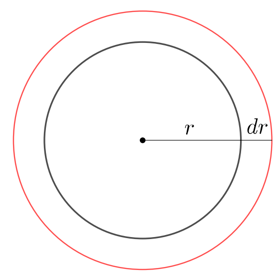
a) Which point is initially moving faster?

b) When, if ever, will the two points have the same velocity?

c) When the clock starts, the two points occupy the same position, but they separate immediately thereafter. Where and when will they next coincide? What will their velocities be then? Will they meet a third time?

56. Molly Bloom throws an object down from the Rock of Gibraltar in such a manner that the distance (in meters) it has fallen after t seconds is given by the function $s = 30t + 4.9t^2$. How fast is the object moving downwards after 5 seconds? What is the object's acceleration then (in m/s^2)?

57. In exercise 50c, you showed that the derivative of a circle's area (with respect to its radius) is its circumference. Is this just a curious coincidence, or is there a deeper reason for it? Thinking geometrically will help you understand.

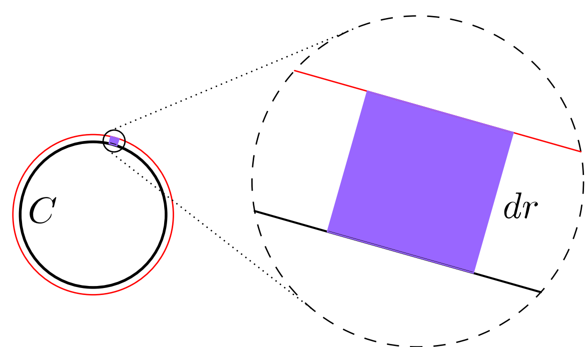


If a circle's radius r increases infinitesimally by dr (necessarily exaggerated in the figure), then its area A will increase infinitesimally by dA. In the figure, dA is the area of the infinitesimally thin ring bounded by the two circles. In this exercise, you'll consider two different geometric explanations of *why* dA/dr equals C, the original circle's circumference.

a) The first explanation is basically computational: Given that the outer circle's radius is $(r + dr)$, express dA, the area in the ring, in terms of r. Then divide by dr, and verify that the derivative dA/dr is indeed $2\pi r$.

b) The second explanation cuts right to the geometric heart of the phenomenon; it doesn't even involve the circle's area and circumference formulas.

Consider the figure at right. The infinitesimally thin ring (whose area is dA) can be broken up into infinitesimal rectangles. Placing all the rectangles end to end, we can construct *one* rectangle, whose height and length will be dr and C respectively, and whose area must therefore be Cdr. However, its area must also be dA (since it was reformed from pieces of the ring).



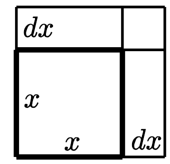
It follows that $dA = Cdr$, which is equivalent to $dA/dr = C$, as claimed.

For many people, this argument provides a flash of insight that renders the formula $dA/dr = C$ obvious. Others, however, are uncomfortable with it, and wonder if it sweeps something important under the rug. Remarkably, both views can coexist in the same mind; one can feel the flash of geometric illumination, and yet still wonder if the means by which it was conveyed are entirely sound.

So, dear reader, did this argument help *you* see why the derivative of a circle's area (with respect to its radius) is its circumference? And is there any of part of the argument that particularly troubles you?

58. Recall that a sphere of radius r has volume $V = (4/3)\pi r^3$. The power rule tells us that $dV/dr = 4\pi r^2$, which is – as the previous problem might lead you to expect – the sphere's surface area. Using analogs of either (or ideally both) of the arguments in the previous problem, try to gain insight into *why* this fact must be true.

59. A square of side length x has area $A = x^2$ and perimeter $P = 4x$. Contrary to what one might expect from the previous two problems, the power rule shows that $dA/dx \neq P$. To understand this geometrically, consider the figure, remembering once again that as with all such schematic depictions of infinitesimals, you must imagine the dx's as being incomparably tinier than they appear.



a) If A represents the original square's area (before the infinitesimal change to x), then to what part of the figure does dA correspond?

b) Use analogs of either (or both) of the arguments from exercise 57 to try to gain insight into why $dA/dx \neq P$.

60. Dippy Dan doesn't know how to communicate mathematical ideas. Collected below are eight samples of his garbled writing. Explain *why* his statements are gibberish, and suggest sensible alternatives that accurately convey what he is presumably trying to express.

a) $\frac{dy}{dx}(x^3) = 3x^2$.

b) If $y = 2x^4$, then $\frac{d}{dx} = 8x^3$.

c) $\frac{d}{dx}(\pi r^2) = 2\pi r$

d) If $y = -7x$, then $dy = -7$.

e) $x^8 = 8x^7$.

f) $2x^2 + 5x^2 \to 7x^2$.

g) $\frac{d}{dx}(3x^4) = 4(3x^3) \to 12x^3$.

h) $d(3x^2) = 6x$.

i) $x^2 \frac{d}{dx} = 2x$.

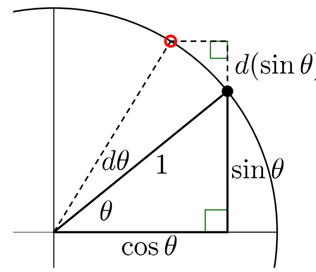
Chapter 2

The Differential Calculus Proper

The Derivatives of Sine and Cosine

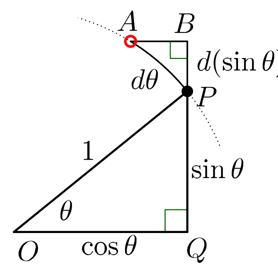
To discover a function's derivative, we change its input infinitesimally, then find the resulting infinitesimal change in output, and finally, we take the ratio of these two infinitesimal changes. Let us do this for sine.

Consider the figure at right, which takes place on the unit circle. By the definition of sine, the sine of θ is the solid point's y-coordinate. Now we'll increase sine's input infinitesimally by $d\theta$ (from θ to $\theta + d\theta$). When we do so, sine's new output will be the *hollow* point's y-coordinate. Thus, the infinitesimal *change* in sine's output, $d(\sin\theta)$, is indeed the length of the segment that I've labelled as such in the figure.



Next, note that *if we measure angles in radians*, the length of the arc between the solid and hollow points must be $d\theta$.[*]

This gives us the picture at right. Note that "arc" AP is *straight*, since $d\theta$ is infinitesimal. Moreover, the infinitesimal right triangle of which it is part is similar to ΔOPQ. [Proof: Any circle is perpendicular to its radii, so $A\hat{P}O$ is a right angle. Thus, $B\hat{P}A$ is $O\hat{P}Q$'s complement, which – as a glance at ΔOPQ shows – is θ. Hence, $\Delta OPQ \sim \Delta PAB$ by AA-similarity.] Consequently, the ratio (leg-adjacent-to-θ):(hypotenuse) is the same in each triangle. That is,



$$\frac{d(\sin\theta)}{d\theta} = \frac{\cos\theta}{1}.$$

Thus we have the derivative we seek: **The derivative of $\sin\theta$ is $\cos\theta$.**

You'll be pleased to know that this intricate argument gives us cosine's derivative as a free bonus. We need only observe in the figure that $AB = -d(\cos\theta)$.[†] Ratios of corresponding parts being equal in similar triangles, we have

$$\frac{-d(\cos\theta)}{d\theta} = \frac{\sin\theta}{1}.$$

Multiplying both sides by -1 reveals that **the derivative of $\cos\theta$ is $-\sin\theta$.**

To sum up, we have proved the following important results. Mark the introductory phrase well!

> When θ is measured in radians,
> $$\boldsymbol{\frac{d}{d\theta}(\sin\theta) = \cos\theta}, \quad \text{and} \quad \boldsymbol{\frac{d}{d\theta}(\cos\theta) = -\sin\theta}.$$

[*] **Proof**: There are 2π radians in a full rotation, so an angle of $d\theta$ radians is $d\theta/2\pi$ of a full rotation. Accordingly, a central angle of $d\theta$ radians is subtended by $d\theta/2\pi$ of the circle's circumference. Since the unit circle's circumference is 2π, the arc length between the solid and hollow points is $d\theta/2\pi$ of 2π, which is indeed $d\theta$, as claimed.

[†] An explanation for the negative: Thinking about cosine's definition reveals that the infinitesimal change $d(\cos\theta)$ and the length AB have the same *magnitude*. Their *signs*, however, are opposite: AB, being a length, is necessarily positive, but since the figure shows that increasing θ by $d\theta$ causes a *decrease* in cosine, $d(\cos\theta)$ must be negative. Hence, $AB = -d(\cos\theta)$.

Exercises.

1. The variables in a function's formula are just placeholders. For instance, $f(x) = x^2$, $y = t^2$, and $h(z) = z^2$ all represent the same underlying *squaring function*, so they all have the same derivative (the *doubling function*), even if we express this fact with different symbols in each case: $f'(x) = 2x$, $dy/dt = 2t$, and $h'(z) = 2z$. This being so, write down the derivatives of the following functions, using notation suitable for each case.

a) $f(\theta) = \sin\theta$ b) $z = \cos t$ c) $x = \sin y$ d) $B = \cos\alpha$ e) $g(w) = \sin w$.

2. Combining this section's results with the derivative's linearity properties, find the derivatives of the following.

a) $f(x) = 3\sin x - 5\cos x$ b) $y = \pi t^2 + \sqrt{2}\sin t + e^5$ c) $g(x) = -\cos x - \sin x$

3. If $y = \cos x$, what is $(y')^2 + y^2$?

4. Pinpoint the precise place in our derivation of sine's derivative where we used radians. Then, by making appropriate adjustments in the rest of the argument, determine the derivative of $\sin\theta$ if one measures angles in *degrees*. You'll find the final result is slightly different (and messier) than when we measure angles in radians.

If calculus did not exist, neither would radians. Mathematicians and scientists adopted radian measure primarily to ensure that sine's derivative is as simple as possible. Any other angle measure (degrees, gradians, or what have you) yields a derivative to which an ugly constant clings like a barnacle.

5. Strictly speaking, the figure and argument in our derivation of sine's derivative covers only the case in which θ lies in the first quadrant. We can easily extend the result to all values of θ, though the details are a bit tedious, which is why I omitted them. If θ lies in the second quadrant, for example, increasing θ causes $\sin\theta$ to *decrease*, with the result that, in the figure, one of the infinitesimal right triangle's sides would be $-d(\sin\theta)$ instead of $d(\sin\theta)$. Draw a picture of this second-quadrant case, and verify that a compensating change in the *other* right triangle ensures that sine's derivative is still cosine. (Then, should you feel so inclined, you can cross the last "t" and dot the final "i" by verifying that the result still holds when θ lies in quadrants three or four.)

6. For very small values of x (where x is measured in radians), $\sin x \approx x$. Explain *why* this approximation, which is frequently used by scientists and engineers, holds. [*Hint: Find the tangent to the graph of $y = \sin x$ at $x = 0$.*]

7. To find the derivative of a composite function such as $y = \sin(5x + 2)$ requires a little trickery. The trick is to rewrite it in terms of its simple components:

$$y = \sin u, \quad \text{where} \quad u = 5x + 2.$$

So far, we've expressed y as a simple function of u, which in turn is a simple function of x. Having accomplished this, it is easy to find dy/dx. We need only make the general observation that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Applied to our particular function, this yields

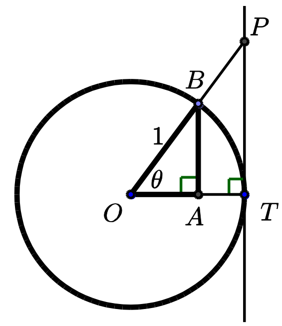
$$\frac{dy}{dx} = (\cos u)5.$$

Substituting $5x + 2$ back in for u (and moving that 5 to its customary location), we have our derivative:

$$\frac{dy}{dx} = 5\cos(5x + 2).$$

Use this trick to find the derivatives of the following functions.

a) $y = \sin(3x + 6)$ b) $y = \cos(3x^2)$ c) $y = \sin(\cos x)$ d) $y = (3x + 1)^{50}$
e) $y = \sin^2 x$ f) $y = 2\sin x \cos x$ [*Hint: A trigonometric identity will help*.]



8. Everyone knows that in the figure at right, $OA = \cos\theta$ and $AB = \sin\theta$. Surprisingly few students (or even teachers) of trigonometry know that the *tangent* function also lives on the unit circle. Its location relative to the unit circle explains why the tangent function is called the tangent function.

a) Using similar triangles and the "SOH CAH TOA" definitions from right-angle trigonometry, prove that $PT = \tan\theta$.

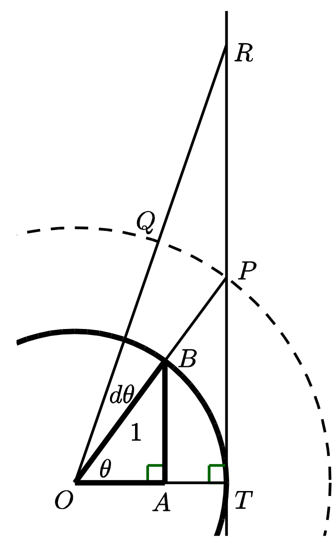
b) The secant function also lives on the unit circle. Prove that $OP = \sec\theta$.

c) The Latin verbs *tangere* and *secare* mean "to touch" and "to cut" respectively. What does this have to do with lines PT, PO, and the circle? (A-ha: Now you know why secant is called secant.)

d) The one trigonometric identity everyone remembers is the "Pythagorean Identity," $\cos^2\theta + \sin^2\theta = 1$. Explain this famous identity's name by thinking about ΔOAB in the figure.

e) There is an alternate version of the Pythagorean identity that often comes in handy in integral calculus: $1 + \tan^2\theta = \sec^2\theta$. Explain how the truth of this identity, too, can be seen in the figure.

f) Those of a more algebraic mindset can derive the alternate Pythagorean identity from the ordinary one by dividing both sides by $\cos^2\theta$. Verify that this is so.



9. Now that you know where $\tan\theta$ and $\sec\theta$ live on the unit circle, we can find their derivatives. (We'll soon learn how to find them algebraically, but doing it geometrically is more aesthetically satisfying.)

a) In the figure at right, the infinitesimal increment $d\theta$ is necessarily exaggerated. Convince yourself that, despite appearances, ΔPQR represents an infinitesimal *right* triangle.

b) Explain why $PR = d(\tan\theta)$.

c) *If we measure angles in radians*, explain why arc QP has length $\sec\theta\, d\theta$.

d) Explain why ΔPQR is similar to ΔOAB.

e) Use this similarity (plus a little algebra) to prove that

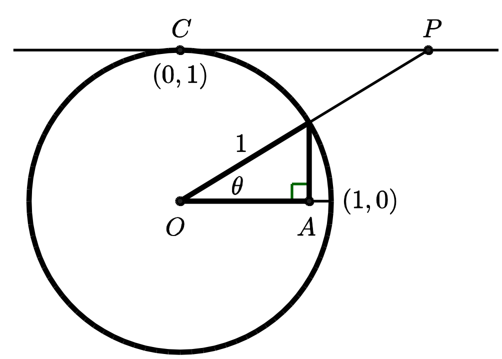
$$\frac{d(\tan\theta)}{d\theta} = \sec^2\theta.$$

f) Explain why $QR = d(\sec\theta)$.

g) Use similar triangles (plus a little algebra) to prove that

$$\frac{d(\sec\theta)}{d\theta} = \sec\theta\tan\theta.$$

10. Poor cotangent and cosecant, the least loved of the six trigonometric ratios, are also denizens of the unit circle.



a) In the figure at right, prove that $PC = \cot\theta$.

b) Prove that $PO = \csc\theta$.

c) Use the figure to explain another alternate version of the Pythagorean identity: $1 + \cot^2\theta = \csc^2\theta$.

d) Explain how to derive the identity in part (c) from the ordinary Pythagorean identity with a little algebra.

e) Using arguments like those you used in exercise 9, discover the derivatives of $\cot\theta$ and $\csc\theta$. [*Hint: You'll need to watch out for negatives, as in the derivation of cosine's derivative above. For instance, you'll need to identify a line segment whose length is* $-d(\cot\theta)$.]

The Product Rule

The derivative of a product is *not* a product of derivatives. (You proved this in exercise 45 of Chapter 1.) Here is the actual rule for a product's derivative.

> **Product Rule.** If u and v are functions of x, then
> $$\boldsymbol{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}.$$

Proof. If we increase the input x by an infinitesimal amount dx, then the outputs of the individual functions u and v change by infinitesimal amounts du and dv. Consequently, the value of their *product* changes from uv to $(u+du)(v+dv)$. With this in mind, we see that

$$\frac{d(uv)}{dx} = \frac{(u+du)(v+dv) - uv}{dx} = \frac{udv + vdu}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}. \quad \blacksquare$$

The product rule is easy to use, as the following examples demonstrate.

Example 1. Find the derivative of $y = x^2 \sin x$.
Solution. Because we can view this function as a product uv, where $u = x^2$ and $v = \sin x$, we can apply the product rule. It tells us that

$$\begin{aligned}\frac{dy}{dx} &= (x^2)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)\\ &= x^2\cos x + 2x\sin x. \quad \blacklozenge\end{aligned}$$

Example 2. Find $\frac{d}{dx}\big((3x^3 + x)\cos x\big)$.
Solution. By the product rule,

$$\begin{aligned}\frac{d}{dx}\big((3x^3+x)\cos x\big) &= (3x^3+x)\frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(3x^3+x)\\ &= -(3x^3+x)\sin x + (9x^2+1)\cos x. \quad \blacklozenge\end{aligned}$$

That's all there is to it.

Exercises.

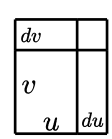
11. Find the derivatives of the following functions.
a) $y = -3x^8\cos x$ b) $y = \sin x \cos x$ c) $y = (9x^4 - x^3 + \pi^2)\sin x$ d) $y = \pi x^2(\sin x + \cos x)$

12. Compute the derivative of $y = (2x^2 + 3x)(5x^2 + 1)$ two different ways, and verify that the results are equal.

13. Derive a "triple product rule" for the derivative of uvw (where u, v, and w are functions of x).

14. Expressed in prime notation, the product rule states that $\big(f(x)g(x)\big)' = \ldots$ *what*?

15. If, for any given x, we can represent the *outputs* of two functions u and v by a rectangle's sides, then we can demonstrate the product rule *geometrically* as follows. Increasing the input by dx yields changes du and dv to the individual outputs, and $d(uv)$ will be represented by an area on the figure. Identify this area, compute it, and divide by dx. You should obtain the product rule.



The Quotient Rule

The quotient rule is uglier than the product rule, but it is just as simple to use.

> **Quotient Rule.** If u and v are functions of x, then
> $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Proof. If we increase the input x by an infinitesimal amount dx, then the outputs of the individual functions u and v change by infinitesimal amounts du and dv. Hence, the value of their *quotient* changes from uv to $(u + du)/(v + dv)$. With this in mind, we see that

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\frac{(u+du)}{(v+dv)} - \frac{u}{v}}{dx} = \frac{\frac{v(u+du) - u(v+dv)}{v(v+dv)}}{dx} = \frac{v(u+du) - u(v+dv)}{v(v+dv)dx} = \frac{vdu - udv}{v^2 dx}.$$

Dividing the last expression's top and bottom by dx yields the expression in the box above. ■

The following idiotic jingle will fix the quotient rule in your memory: *Low dee-high minus high dee-low, over the square of what's below*. ("High" and "low" being the top and bottom functions in the quotient, while "dee" indicates "the derivative of".) Recite it when using the quotient rule, and all will be well.

In exercise 9, we employed a devilishly clever geometric argument to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$. With the quotient rule's help, we can prove this fact mechanically, obviating the need for cleverness.

Example 1. Prove that $\frac{d}{dx}(\tan x) = \sec^2 x$.

Solution. Thanks to a well-known trigonometric identity for tangent, we have

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{(\cos x)\frac{d}{dx}(\sin x) - (\sin x)\frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{(by the quotient rule)}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad \text{(by basic trig identities).} \ \blacklozenge$$

The power rule tells us that to take a power function's derivative, we simply reduce its power by 1, and multiply by the old power. (That is, $(x^n)' = nx^{n-1}$.) So far, we've only proved that this holds when the power is a positive integer. With the quotient rule's help, we can show that it holds for *all* integers.

Example 2. Prove that the power rule holds for all integer powers.

Solution. We've already established this for positive integers, and it obviously holds when the power is zero. (Be sure you see why.) To finish the proof, let us suppose $-m$ is a *negative* integer. Then m is a positive integer, so the derivative of x^m is mx^{m-1}. Bearing this in mind, we find that

$$\frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m\frac{d}{dx}(1) - 1\frac{d}{dx}(x^m)}{x^{2m}} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1},$$

which shows that the power rule holds even when the power is a negative integer. ♦

Many problems require us to combine different derivative rules. This is easy enough if you write out the intermediate steps.

Example 3. Find the derivative of $y = \frac{x^2 \sin x}{1+x^2}$.

Solution. $\frac{dy}{dx} = \frac{(1+x^2)\frac{d}{dx}(x^2 \sin x) - (x^2 \sin x)\frac{d}{dx}(x^2)}{(1+x^2)^2}$ (quotient rule)

$= \frac{(1+x^2)(2x \sin x + x^2 \cos x) - 2x^3 \sin x}{(1+x^2)^2}$ (product rule). ♦

So much for the quotient rule.

Exercises.

16. Expressed in prime notation, the quotient rule states that $\left(\frac{f(x)}{g(x)}\right)' =$ … *what*?

17. In example 1, we proved that $(\tan x)' = \sec^2 x$. To complete our list of trig functions' derivatives, prove that

a) $(\sec x)' = \sec x \tan x$. b) $(\csc x)' = -\csc x \cot x$. c) $(\cot x)' = -\csc^2 x$.

18. Now that you've found the derivatives of all six trigonometric functions, memorize them. There are patterns (particularly among cofunction pairs) that will make this easier if you notice them. Notice them.

19. Find the derivatives of the following functions (simplifying your answers when possible.)

a) $y = \frac{2}{1-x}$ b) $y = \frac{3x^3}{\cos x}$ c) $y = \frac{x+\pi^2}{\tan x}$ d) $y = \frac{x^2}{3x+5}$ e) $y = \frac{4x \sin x}{2x + \cos x}$ f) $y = \frac{x^2 \sin x}{x \sec x}$

g) $y = \frac{1}{x}(\sec x + \tan x) + \ln 2$ h) $y = x^{-2} - x^{-3} + x^{-4} - x^{-5}$

20. In Example 2, we showed that the power rule holds for all integer exponents. In this exercise, you'll prove that the power rule holds for all *rational* exponents. This will require several steps.

a) First, you'll establish the case where the exponent is a "unit fraction" (i.e. of the form $1/n$ for an integer n). If $y = x^{1/n}$, you can find dy/dx with a diabolical trick: find dx/dy instead and then take its reciprocal! Your problem: Do this.

[*Hint: Begin by expressing x as a function of y. Then compute dx/dy, take its reciprocal, and rewrite it in terms of x. After a little algebra, you should be able to establish that $dy/dx = (1/n)x^{(1/n)-1}$, as claimed*.]

b) Next, having handled the case of a unit fraction exponent, we'll tackle the general rational exponent m/n. The key is to rewrite $y = x^{m/n}$ as the composition of simpler functions whose derivatives we already know. Namely, we can write $y = u^m$, where $u = x^{1/n}$.

Use the trick from exercise 7 (and some algebra) to prove that $dy/dx = (m/n)x^{(m/n)-1}$.

21. Explain how it follows from the previous exercise that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$.

22. Derivatives of square roots occur often enough to warrant memorizing the formula in the previous exercise. Doing so means that you won't have to rewrite square roots in terms of exponents and apply the power rule each time you need a square root's derivative. After committing this useful formula to memory, find the derivatives of the following functions. (Express your final answers without fractional exponents.)

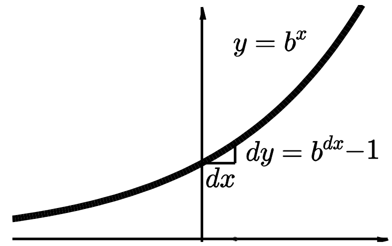
a) $y = \sqrt{x} \sin x$ b) $y = \sqrt{9x} + \sqrt{x}$ c) $y = \frac{\sqrt{x}}{\tan x}$ d) $y = 3\sqrt[3]{x^5}$ e) $y = \csc(\sqrt{x})$

23. On the graph of $y = \sin x$, consider two tangent lines: one where $x = 0$, and another where $x = \pi/6$. At which point do these two lines cross?

Derivatives of Exponential Functions

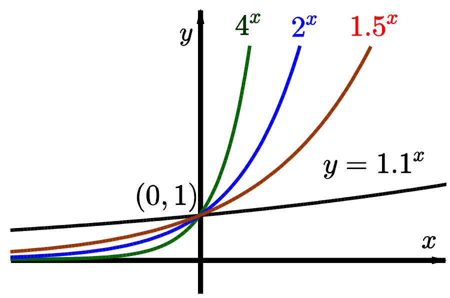
Our quest for the derivative of the exponential function b^x (for any base b) begins algebraically:

$$\frac{d(b^x)}{dx} = \frac{b^{x+dx} - b^x}{dx} = \frac{b^x b^{dx} - b^x}{dx} = \left(\frac{b^{dx}-1}{dx}\right) b^x.$$



The expression in parentheses is, as the figure shows, the slope of b^x at $(0,1)$. Consequently, we can rewrite the preceding equation as

$$\frac{d(b^x)}{dx} = \begin{pmatrix}\text{The slope of } b^x \\ \text{at point } (0,1)\end{pmatrix} b^x.^*$$



As the next figure demonstrates, increasing the base b increases the slope. Moreover, *some* base (between 1.1 and 4) will make the slope at $(0,1)$ equal to 1 exactly. We call this special base $\boldsymbol{e}$. This endows the function e^x with a truly remarkable property:

$$\frac{d}{dx}(e^x) = (1)e^x = e^x.$$

That is, the function e^x is its own derivative.

No doubt you are wondering if this is the same e you've met before as the natural logarithm's base. It is. In precalculus, e is enigmatic. In calculus, e arises *naturally*, since, by its very definition, it is the base of the one exponential function equal to its own derivative.† Perhaps e should not be introduced in precalculus classes at all; grasping e's significance before knowing what a derivative is may be as hopeless as understanding π's significance before knowing what a circle is.

Now that we know the derivative of e^x, we can find the derivative of b^x for *any* base b. We begin by using logarithmic properties to convert an arbitrary-based exponential function to one with base e.

$$b^x = \left(e^{\ln b}\right)^x = e^{(\ln b)x}.$$

Next, we'll use a trick that you've already used in several exercises (see exercises 7, 20b, and 22e). Namely, we shall rewrite b^x as a composition of simpler functions whose derivatives we already know.

$$b^x = e^u, \quad \text{where} \quad u = (\ln b)x.$$

This allows us to compute the derivative we seek:

$$\frac{d(b^x)}{dx} = \frac{d(b^x)}{du}\frac{du}{dx} = (e^u)(\ln b) = (\ln b)e^{(\ln b)x} = (\ln b)\left(e^{(\ln b)}\right)^x = \boldsymbol{(\ln b) b^x}.$$

* For example, at $(0,1)$, the graph of $y = 2^x$ has a slope of approximately 0.7, so it follows that $\frac{d}{dx}(2^x) \approx 0.7(2^x)$.

† Stated with a bit more care, we can define e is the base of the only function **of the form $\boldsymbol{b^x}$** that is equal to its own derivative. It is reasonable to ask if there any *other* functions (of another form) that equal their own derivatives. It turns out that e^x is essentially unique in this regard; the only other such functions that have this property are constant multiples of e^x.

We have thus established the following results:

Derivatives of Exponential Functions.

$$\frac{d}{dx}(e^x) = e^x.$$

More generally, for any positive base b,

$$\frac{d}{dx}(b^x) = (\ln b)b^x.$$

We saw earlier (in exercise 4) that we use radians precisely because they simplify sine's derivative; if calculus didn't exist, neither would radians. Similarly, we use e as a base for exponential functions and logarithms because it simplifies their derivatives. If calculus did not exist, no one would bother with e. Calculus forces strange things up from the depths.

When exponential functions are present, their inverses – logarithms – are never far away. Once we know a function's derivative, we can find its inverse's derivative (by using the trick from exercise 20a). I'll use that trick here to find the natural logarithm's derivative. (And you'll use the trick in the exercises to find the derivatives of the base-10 logarithm and the inverse trigonometric functions.)

Problem. Find the derivative of $y = \ln x$.

Solution. Since $y = \ln x$ implies $x = e^y$, we have $\frac{dx}{dy} = e^y$. Thus, $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$. ♦

The natural logarithm's derivative is important. Commit it to memory.

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}.}$$

Exercises.

24. Find the following functions' derivatives.

a) $y = e^x \sin x$ b) $y = \frac{\ln x}{3 \tan x}$ c) $y = -2^x \cos x + 3x^2 \ln x$ d) $f(x) = e^x\sqrt{x} + e^2$

e) $g(x) = 2^x 3^x 4^x 5^{x+1}$ [*Hint: Some preliminary algebra will help*.] f) $y = \csc x \sin x - \frac{\ln e^x}{x}$

g) $w = \sqrt[3]{t^5} \sec t + \frac{1}{t}$ h) $V = \frac{1}{y} + \ln y$ i) $k(x) = \frac{2^x \cos^2 x + 2^x \sin^2 x}{\sqrt{x}}$ j) $y = x^e/e^x$

25. Does the graph of $y = 2^x - x$ have a horizontal tangent at any point? If so, find that point's x-coordinate.

26. Find the derivative of $y = \log_{10} x$ by adapting the trick we used to establish the natural logarithm's derivative.

27. The same trick can be used to find the derivative of $y = \sin^{-1} x$, but a trigonometric twist complicates the end.

a) If you solved the previous problem, you'll find it easy to show that $dy/dx = 1/\cos(\sin^{-1} x)$. Do so.

b) We can simplify the ghastly expression $\cos(\sin^{-1} x)$ with a trigonometric identity. To derive it, first explain why $\cos(\sin^{-1} x) = \pm\sqrt{1 - \sin^2(\sin^{-1} x)}$. Then simplify this and explain why the $\pm$ must in fact be a plus. [*Hint for the* ± business: *Think about the range of inverse sine, and what cosine does to values in that range.*]

c) Conclude that $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

28. By shadowing the argument we used in the previous exercise, show that $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.

29. Finally, show that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. [*Hint: Use the alternate Pythagorean identity*, $\sec^2 x = 1 + \tan^2 x$.]

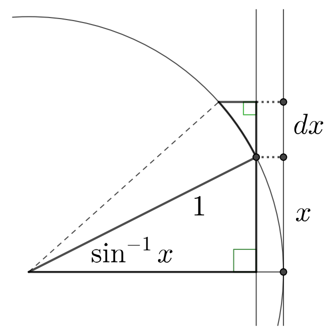
30. In this problem, you'll learn how to approximate the numerical value of e.

a) Explain how e's definition implies that $(e^{dx}-1)/dx = 1$, which, in turn, implies that $e = (1+dx)^{1/dx}$. [*Hint: Express the slope of the graph of* $y = e^x$ *at* $(0,1)$ *two different ways. Equate the results*.]

b) The formula for e in part (a) is approximately true if we replace the infinitesimal dx by a small *real* value Δx. The smaller the value of Δx, the better the approximation. Using a calculator, substitute small values of Δx into the approximation $e \approx (1+\Delta x)^{1/\Delta x}$, and convince yourself that $e \approx 2.71828$.

c) One way to make Δx small is to let $\Delta x = 1/n$, where n is a large whole number. If we make this substitution, then we can reformulate the approximation in the previous part as follows: $e \approx (1+1/n)^n$, where n is a large whole number; the larger n is, the better the approximation. Using a calculator, substitute some large values of n into this approximation to confirm what you discovered in part (b).

31. In #27, some clever algebraic shenanigans led you to *inverse sine*'s derivative. Since inverse sine, however, is an essentially geometric function, geometrically-minded souls will find the following geometric derivation more illuminating.



a) Defining x as in the figure, explain why the angle marked $\sin^{-1} x$ has been marked appropriately.

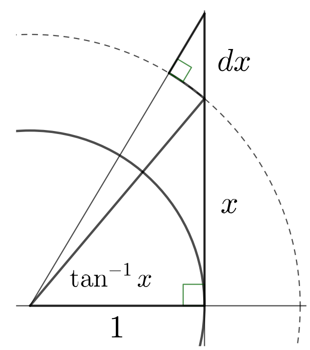
b) Increasing x by an infinitesimal amount dx induces an infinitesimal change in $\sin^{-1} x$. Locate $d(\sin^{-1} x)$ on the figure.

c) One leg of the figure's infinitesimal triangle is in fact an arc. Find its length.

d) Explain why the two right triangles emphasized in the figure are similar.

e) Use this similarity to show that $d(\sin^{-1} x)/dx = 1/\sqrt{1-x^2}$.

32. In light of the previous problem, it should come as no surprise that the other inverse trigonometric functions' derivatives can be justified geometrically. In this problem, you'll carry this out for inverse tangent.



a) Explain why the angle marked $\tan^{-1} x$ has been appropriately marked.

b) Locate $d(\tan^{-1} x)$ on the figure.

c) Use the Pythagorean theorem to find the large right triangle's hypotenuse.

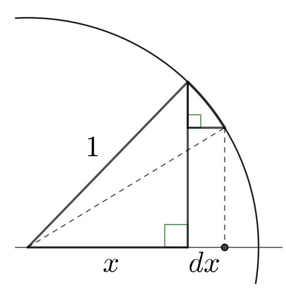
d) One leg of the figure's infinitesimal triangle is in fact an arc. Find its length.

e) Explain why the two right triangles emphasized in the figure are similar.

f) Use this similarity to show that

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}.$$

33. The geometric derivation of inverse cosine's derivative require a bit more care: One must avoid traffic jams in the figure while keeping a watchful eye on negatives. Apart from that, though, it is business as usual.



a) Defining x as in the figure, locate $\cos^{-1} x$.

b) Observe that increasing x by an infinitesimal amount dx induces an infinitesimal *decrease* in $\cos^{-1} x$, so $d(\cos^{-1} x)$ is negative. Locate the quantity $-d(\cos^{-1} x)$, which is *positive*, on the figure.

c) Complete the argument.

The Chain Rule

"He hath hedged me about, that I cannot get out: he hath made my chain heavy."
- Lamentations 3:7

You have already met the chain rule, for I smuggled it – incognito – into some earlier exercises (7, 20b). The rule is really nothing but the trivial observation that we can factor dy/dx as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

To find derivatives of *composite functions* (i.e. functions of functions, such as $y = \sin(\ln x)$), we just call the composite's "inner function" u, then grind out the two derivatives on the right-hand side above. Note that the first factor, dy/du, will be a function of u, so after computing it, we must rewrite it in terms of x, which is a simple matter of substituting the inner function itself for u wherever it appears.

Example 1. Find the derivative of $y = \sin(\ln x)$.

Solution. If we rewrite this function as $y = \sin u$, where $u = \ln x$, the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \left(\frac{1}{x}\right) = \frac{\cos(\ln x)}{x}. \quad \blacklozenge$$

Note that in our final expression, we've eliminated all traces of u.

Example 2. Find the derivative of $y = (3x^2 - 1)^{10}$.

Solution. If we rewrite this as $y = u^{10}$, where $u = 3x^2 - 1$, the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 10u^9(6x) = 60x(3x^2 - 1)^9. \quad \blacklozenge$$

We can speed up the chain rule procedure (and make it independent of Leibniz notation) by restating it in words. To do so, we first note that dy/du and du/dx are the derivatives of the composite's outer and inner functions, respectively. We can now state the "fast version" of the chain rule: **Take the outer function's derivative, *evaluate it at the inner function*, then multiply by the inner function's derivative**. (The italicized step corresponds to eliminating u from the final expression.)

Here's an example of the fast version in action.

Example 3. Find the derivative of $y = \tan(2x^3 + 3x^2 + x + 1)$.

Solution. By the chain rule, $\frac{dy}{dx} = \underbrace{\left(\sec^2(2x^3 + 3x^2 + x + 1)\right)}_{\substack{\text{Derivative of the outer} \\ \text{evaluated at the inner.}}} \underbrace{(6x^2 + 6x + 1)}_{\substack{\text{Derivative} \\ \text{of the inner}}}$ ♦

This fast version of the chain rule is especially useful when the chain rule must be combined with the product rule, the quotient rule, or even a second instance of the chain rule.

Example 4. Find the derivative of $y = \sqrt{3x^2 \sin x}$.

Solution.
$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2\sqrt{3x^2 \sin x}} \cdot \frac{d}{dx}(3x^2 \sin x) && \text{(chain rule)} \\ &= \frac{1}{2\sqrt{3x^2 \sin x}} [6x \sin x + 3x^2 \cos x] && \text{(product rule)} \quad \blacklozenge \end{aligned}$$

For our grand finale, a fourfold composition of functions. It looks more complicated, but isn't really; we just follow the chain of derivatives, link by link, from the outermost to the innermost function.

Example 5. Find the derivative of $y = \sin(\tan(\ln(2x+2)))$.

Solution.
$$\begin{aligned}\frac{dy}{dx} &= \cos(\tan(\ln(2x+2)))\frac{d}{dx}(\tan(\ln(2x+2))) && \text{(chain rule)}\\ &= \cos(\tan(\ln(2x+2)))\,(\sec^2(\ln(2x+2))\frac{d}{dx}(\ln(2x+2)) && \text{(chain rule)}\\ &= \cos(\tan(\ln(2x+2)))\,(\sec^2(\ln(2x+2))\left(\frac{1}{2x+2}\right)\frac{d}{dx}(2x+2) && \text{(chain rule)}\\ &= \cos(\tan(\ln(2x+2)))\,(\sec^2(\ln(2x+2))\left(\frac{1}{x+1}\right). && \blacklozenge\end{aligned}$$

Mastering the chain rule requires practice. Behold your opportunity to obtain it:

Exercises.

34. Express the chain rule in prime notation.

35. Find the derivatives of the following functions.

a) $y = e^{5x}$ b) $y = \cos(3x^2) + 1$ c) $y = \ln(\ln x)$ d) $y = (2x^3 - x)^8$ e) $y = 2^{\tan x}$

f) $y = \frac{1}{(3x-4)^2}$ g) $y = x\sqrt{144 - x^2}$ h) $y = e^{-x^2}$ i) $y = \sqrt[3]{x + x^3}$ j) $y = \cos^2 x$

k) $y = (1 - 4x^3)^{-2}$ l) $y = \cos(\csc(1/x))$ m) $y = e^{\sin^2 x}$ n) $y = \ln(\sin(\ln x))$ o) $y = \frac{e^{\sin x}}{\sqrt{\sec x}}$

p) $y = \ln\left(\frac{x^2+4}{2x+3}\right)$ q) $y = 10^{-x/(1+x^2)} + 2^{\pi}$ r) $y = 5x^2 e^{6x^2+e}$ s) $y = \ln\left(\sqrt[3]{6x^2 + 3x}\right)$ t) $y = e^{\pi^2}$

36. The 27th-degree polynomial $y = (2x^3 - x^2 + x + 1)^9$ crosses the y-axis at some point. Consider the tangent to the polynomial's graph at this point. Find the point at which this tangent line crosses the x-axis.

37. Does the graph of $y = \ln(\sin(\ln x))$ have a horizontal tangent line at any of its points? If so, at how many points? Find the exact coordinates of one such point.

38. Let $\sin_{<n>}(x)$ denote n nested sine functions, so that, for example, $\sin_{<3>}(x) = \sin(\sin(\sin x))$.

a) Where does the graph of $y = \sin_{<2>}(x)$ cross the x-axis?

b) What is the range of $y = \sin_{<2>}(x)$? State it exactly, then use a calculator to approximate its endpoints.

c) What is the slope of $y = \sin_{<2>}(x)$ at the origin? At other points where the graph crosses the x-axis?

d) Repeat the first three parts of this problem, but with $y = \sin_{<3>}(x)$.

e) Graph $y = \sin_{<2>}(x)$ and $y = \sin_{<3>}(x)$ as best you can. Then try to guess what happens to the graphs of the functions $y = \sin_{<n>}(x)$ for larger and larger values of n.

Part II
Integral Calculus

Chapter 4
The Basic Ideas

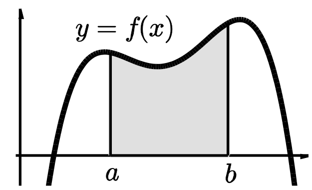
Integrals: Intuition and Notation

"We could, of course, use any notation we want; do not laugh at notations; invent them, they are powerful.
In fact, mathematics is, to a large extent, the invention of better notations."

- Richard Feynman, *Lectures on Physics*, Chapter 17, Section 5.

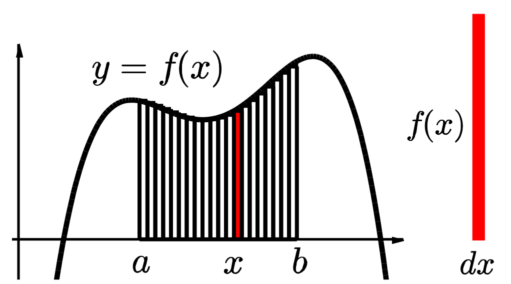
In principle, finding the area of a polygon (a figure with straight boundaries) is easy: We just chop it into triangles, find their areas, and add them up. What could be simpler? Even the village idiot knows the formula for a triangle's area, and many a bright twelve year old can explain *why* it holds, too.*

Finding the area of a region with a *curved* boundary is trickier, since we can't chop it into a finite number of "nice" pieces. (Here, "nice" just means having a shape whose area we can find.) We can, however, chop it into *infinitely* many nice pieces by making each piece *infinitesimally* thin. For example, we can (mentally) chop the shaded region at right into infinitely many infinitesimally thin rectangles, as the schematic figure below suggests. (Naturally, you must imagine *infinitely many* rectangles there, not just the twenty I have actually drawn.) If we can somehow find these rectangles' areas and sum them up, we will have the region's area.



This idea sounds promising in theory, but can we sum infinitely many infinitesimal areas in practice? Thanks to the magic of integral calculus, the answer is *yes*. The magic's deep source, which you will soon behold, is a result called **The Fundamental Theorem of Calculus**. But before you can understand the Fundamental Theorem, you must, as any adept of the dark arts will appreciate, be initiated into the integral calculus's *symbolic* mysteries.

Consider a typical infinitesimal rectangle standing at a typical point, x. Since its height is $f(x)$ and its width is dx (an infinitesimal bit of x), the rectangle's area is $f(x)dx$. The whole region's area is an infinite sum of these $f(x)dx$'s: one for each possible position of x from a to b. To indicate an infinite sum of this sort, we use Leibniz's elegant integral symbol, $\int$, which he intended to suggest a stretched out "S" (for "Sum"). Using this notation, we express the whole region's area as follows:



$$\int_a^b f(x)dx.$$

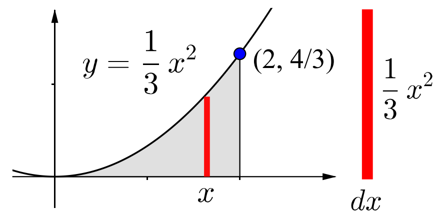
The values a and b near the top and bottom of the integral sign are the **boundaries of integration**, the values of x at which we start and stop summing up the $f(x)dx$'s.

Half the task of learning integral calculus is developing an intuitive feel for this notation and learning to think in terms of infinite sums of infinitesimals. The subject's other half is about *evaluating* integrals – determining their actual numerical values. To evaluate an integral, we'll need a single beautiful theorem (The Fundamental Theorem of Calculus) plus various technical tricks by means of which we'll lure recalcitrant integrals into the Fundamental Theorem's sphere of influence.

* If, being older than twelve, you need a reminder, see exercise 1.

Example 1. Express the area of the shaded region in the figure as an integral.

Solution. We begin by thinking of the region as an infinite collection of infinitesimally thin rectangles. A typical rectangle (at a typical point x, as highlighted in the figure) has height $x^2/3$ and width dx, so its area is $(x^2/3)dx$.



Hence, the integral

$$\int_0^2 \frac{x^2}{3}\,dx.$$

represents the shaded region's entire area. ♦

Once you've learned the Fundamental Theorem of Calculus, you'll be able to evaluate such integrals. But first, let's get comfortable interpreting and using integral notation. In the following examples, the goal is to understand *why* the integrals that eventually appear represent the quantities I claim they represent.

Example 2. Suppose that t hours after noon, a car's speed is $s(t)$ miles per hour. Express the distance that it travels between $2{:}00$ and $4{:}00$ pm as an integral.

Solution. For an object moving at constant speed, we know that $distance = speed \times time$.* Of course, during a typical two-hour drive, the car's speed is not constant; it can change wildly even within a single minute. Within a *second*, however, the car's speed can change only a bit. Within a tenth of a second, it can change still less, and within a hundredth of a second, its speed is nearly constant. As true calculus masters, we push this trend to extremes, recognizing that within any given *instant* (i.e. over an *infinitesimal* period of time), the car's speed actually *is* constant.

We'll now apply this crucial insight. At any time t, the car's speed is $s(t)$ *and remains so for the next instant*, so during that instant (whose duration we'll call dt), the car travels $\boldsymbol{s(t)dt}$ miles.

The distance that the car travels between $2{:}00$ and $4{:}00$ is the sum of the distances it travels during the (infinitely many) instants between those times – the sum, that is, of all of the $\boldsymbol{s(t)dt}$'s for all values of t between 2 and 4. Hence, the integral

$$\int_2^4 s(t)\,dt.$$

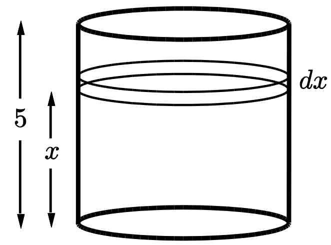
represents the total distance (in miles) that the car travels between $2{:}00$ and $4{:}00$ pm. ♦

Such is the spirit of integral calculus: We mentally shatter a function (whose values are *globally variable*) into infinitely many pieces, each of which is *locally constant* on an infinitesimal part of the function's domain. These constant bits are easy to analyze; having analyzed a typical one, we sum up the results with an integral, which the Fundamental Theorem will soon let us evaluate.

The overall process is thus one of *dis*integration (into infinitesimals) followed by *re*integration (summing up the infinitesimals) so as to reconstitute the original whole in a profoundly new form.

* E.g. A car driving at a constant speed of 60 mph for 2 hours would cover 120 miles.

Example 3. In physics, the **work** needed to raise an object of weight W through a distance D (against the force of gravity) is defined to be WD. Suppose that a topless cylindrical tank, 5 feet high, with a base radius of 2 feet, is entirely full of calculus students' tears. Find an expression for the work required to pump all the salty liquid over the tank's top. (The density of the tears, by the by, is 64 pounds per cubic foot.)



Solution. Different parts of the liquid must be raised different distances. This *seems* to complicate our work (pun intended), but if we simply observe that all the liquid at the same height must be raised the same distance, an integral will save us. We simply don our calculus glasses and view the cylinder as a stack of infinitesimally thin cylindrical slabs (as though it were a roll of infinitely many infinitesimally thin coins). A typical slab at height x must be lifted through a *distance* of $5 - x$ feet. Since its *weight* is $256\pi dx$ lbs (64 times its volume of $\pi(2^2)dx = 4\pi dx$ cubic feet), the *work* needed to lift it to the tank's rim is $\boldsymbol{(5 - x)256\pi dx}$ foot-lbs.

To find the total amount of work required to remove all the slabs, we simply add up these $(5 - x)256\pi dx$'s as x runs from 0 (the tank's bottom) to 5 (the tank's top). Thus,

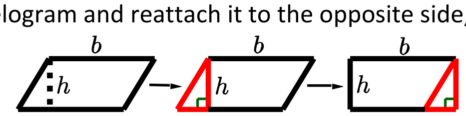
$$\int_0^5 (5 - x)256\pi \, dx.$$

is the total amount of work (in foot-lbs.) required to lift all the tears over the rim. ♦

Exercises.

1. Everyone knows that a rectangle with base b and height h has area bh.
Less well-known is that a *parallelogram* with base b and height h also has area bh.*

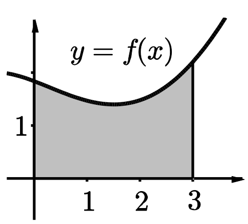
(Proof: In the figure at right, we cut a triangle away from a parallelogram and reattach it to the opposite side, turning the parallelogram into a *rectangle*. Since area was neither created nor destroyed during the operation, the original parallelogram's area must equal that of the rectangle, which is, of course, bh.)



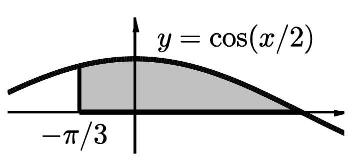
Your problem: Explain why **every triangle's area is *half* the product of its base and height**. [*Hint: We established the parallelogram's area formula by relating the parallelogram to a shape whose area we already knew. Use the same trick to establish the area formula for a triangle.*]

2. Express the following areas as integrals. In each case, sketch a typical infinitesimal rectangle.

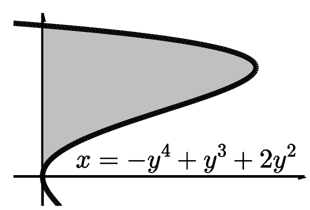
a)



b)



c)



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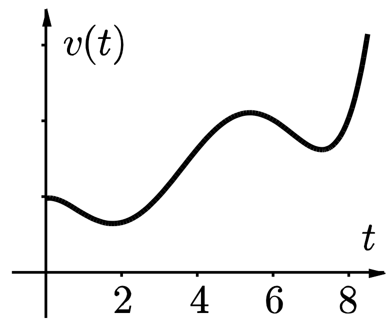
* A parallelogram's "height" is the distance between its parallel bases, *not* the length of its sloped sides.

3. Suppose the velocity of a particle moving along a straight line is, after t seconds, $v(t)$ meters per second.

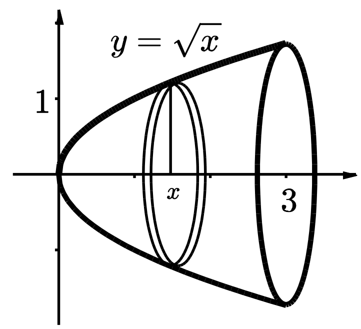
a) Based on the graph at right, does the particle ever reverse its direction?

b) Can the total distance traveled by the particle between $t = 4$ and $t = 8$ seconds be expressed as an integral? If not, why not? If so, write down the integral that does the job, and explain *why* it represents this distance.

c) Observe that the integral you've produced also represents the area of the region that lies below the curve, above the x-axis, and between the vertical lines $t = 4$ and $t = 8$. The strange moral of this story: An area can sometimes represent a distance.



4. Draw the graph of $y = \sqrt{x}$ restricted to $0 \leq x \leq 3$. Look at your drawing and imagine revolving the graph around the x-axis. This will generate the three-dimensional "solid of revolution" shown at right. We can express its *volume* as an integral as follows. First, imagining the figure as solid, we mentally chop it up into infinitely many infinitesimally thin slices as though it were a loaf of bread. A schematic representation of one such slice is shown in the figure.



a) Explain why the solid's cross-sections will be circles.

b) The radii of these circles can vary from 0 to $\sqrt{3}$, depending on the point at which we take our slice. The closer the two circular cross-sections are to one another in space, the closer the lengths of their radii will be. When two circular cross-sections are *infinitesimally close* to one another, then their radii are effectively equal. Consequently, we can conceive of a typical infinitesimally thin slice as being an infinitesimally thin *cylinder*. This is excellent news because we know how to find a cylinder's volume: We multiply the area of its circular base by its height. In the specific example at hand, convince yourself that a typical slice of our solid (taken at a variable point x) is a cylinder whose circular base has radius $\sqrt{x}$ and whose height is dx (the slice's infinitesimal thickness). Then write down an expression for the volume of a typical slice at x.

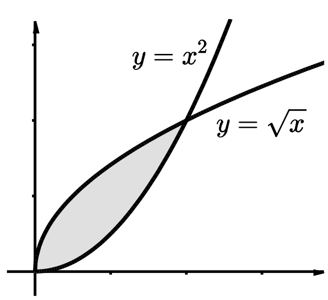
c) Use your result from the previous part to express the volume of the entire solid as an integral.

d) If we revolve one arch of the sine wave around the x-axis, find an integral that represents the volume of the resulting solid. Draw pictures and let them guide you to the integral.

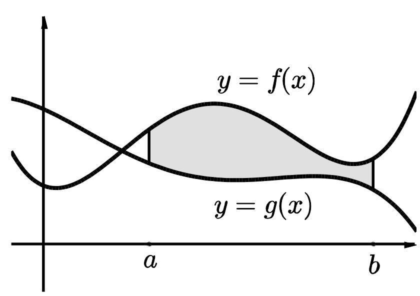
e) If we revolve half an arch of the sine wave (from 0 to $\pi/2$) around the $\boldsymbol{y}$-axis, draw a picture of the resulting solid, and write down an integral that represents its volume.

5. Express the shaded areas as integrals. [*Hint: The usual story. Disintegrate each area into infinitesimally thin rectangular slices, find an expression for the area of a typical slice, then integrate.*]

a)



b)

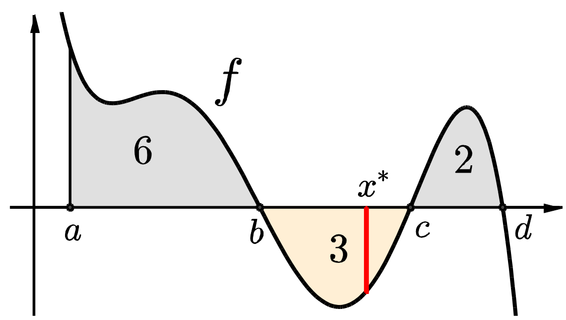


Positives and Negatives

"Natural selection can act only by the preservation and accumulation of infinitesimally small inherited modifications..."
- Charles Darwin, *On The Origin of Species*, Chapter IV.

Integrals always have the form $\int_a^b f(x)dx$. To interpret an integral, it sometimes helps to imagine an obscure mythological creature, the Integration Demon, who traverses the number line from a to b, pen and ledger in hand; at each real number in that interval, he evaluates f and multiplies the result by dx, yielding $f(x)dx$. His diabolical task is not just to find all the individual $f(x)dx$'s, but to *add them all up*. Their grand total is the integral's value.

When we integrate in the usual direction (left to right on the number line), dx is always a *positive* infinitesimal change in x.* Thus, for any x at which $f(x)$ happens to be *negative*, $f(x)dx$ is negative too. Naturally, a negative $f(x)dx$ goes, so to speak, into the debit column of the integration demon's ledger. If, at the end of the day, the debits outweigh the credits, then the integral's value will be negative.



Example 1. In the figure, it's easy to see that

$$\int_a^b f(x)dx = 6.$$

But what about the integral of f from b to c? Between b to c, each $f(x)dx$ will be *negative* since $f(x) < 0$ throughout that entire interval. Although the negative $f(x)dx$'s obviously can't represent areas (which are necessarily positive), they are still related to areas in a simple way.

Consider the infinitesimal rectangle shown at x^*. Its height is $-f(x^*)$, so its area is $-f(x^*)dx$. This means that $f(x^*)dx$ must be the *negative* of that rectangle's area. Accordingly, as the integration demon goes from b to c, he'll be summing up not the areas of infinitesimal rectangles, but rather the *negatives* of their areas. Since these rectangles fill a region whose total area is 3, the demon's grand sum of all the $f(x)dx$'s between b to c must therefore be -3. That is,

$$\int_b^c f(x)dx = -3.$$

Next, suppose we wish to integrate from a to c. This integral's value is obvious if we think of the integration demon and his ledger. As he proceeds from a to b, he adds up lots of $f(x)dx$'s; their sum is 6 (the area of the figure's first shaded region). As he travels from b to c, he tallies up still more $f(x)dx$'s. Their sum, as discussed above, is -3, which brings his grand total down from 6 to 3. Consequently,

$$\int_a^c f(x)dx = 3. \qquad \blacklozenge$$

Please be certain that you understand the ideas that justify the preceding example's conclusions. If you do, then read on. If not, then go back, reread and think about them until you do understand.

* One *can* integrate "backwards" (*i.e.* right to left on the number line), but in practice, one doesn't. Should sheer perversity drive someone to do so, then his dx's would be *negative* infinitesimal changes. More on this in exercise 27.

Negative $f(x)dx$'s in an integral are perfectly acceptable in physical contexts, too, as we'll now see.

Example 2. A fly enters a room and buzzes around erratically, greatly irritating everyone therein. We'll concentrate exclusively on the fly's motion in the up/down dimension. Let $v(t)$ be the fly's upwards velocity after t seconds in the room. (Thus, $v(5) = -2$ m/s would signify that 5 seconds after entering the room, the fly's height is *decreasing* at a speed of 2 meters per second.)

Over a mere *instant* (i.e. infinitesimal bit of time), the fly's velocity is effectively constant. Hence, its *speed*, $|v(t)|$, is constant during the instant too, which means that during the instant, the constant-speed formula $speed \times time = distance$ applies. Thus, $|v(t)|dt$ represents the distance that the fly travels in the instant following time t. For example, the statement

$$\int_0^{60} |v(t)|dt = 10$$

tells us that during its first minute in the room, the fly travels a total of 10 vertical meters, sometimes going up, sometimes going down.

Whereas $|v(t)|dt$ is always positive and represents the distance that the fly travels in a given instant, the quantity $v(t)dt$ can be negative; it therefore indicates not only the distance travelled by the fly in a given instant, but the fly's *direction* during that instant as well. Positive $v(t)dt$'s correspond to altitude gains, while negative $v(t)dt$'s correspond to losses. It follows that when we sum up all the $v(t)dt$'s over some interval of time, we will end up with the fly's net gain in altitude over that time period, which could, of course, be negative. (Such a net change in an object's position is called its *displacement*.) Consequently, the equation

$$\int_0^{60} v(t)dt = -0.5$$

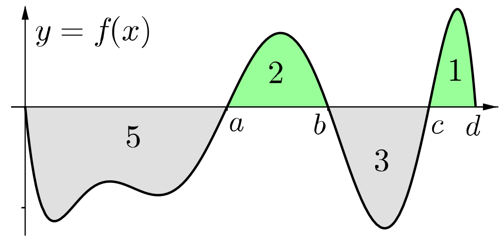
tells us that after sixty seconds of buzzing around (during which, according to the previous integral, it travelled a total of ten vertical meters) the fly's height was half a meter lower than it was when it first entered the room. ♦

After a few exercises, you'll be ready to begin climbing towards the Fundamental Theorem of Calculus, which will allow you to begin evaluating integrals at last.

Exercises.

6. In Example 1 above, explain why $\int_a^d f(x)dx = 5$.

7. The numbers in the figure at right represent areas. Evaluate the following integrals.

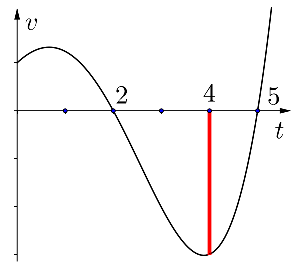


a) $\int_0^a f(x)dx$ b) $\int_a^b f(x)dx$ c) $\int_0^b f(x)dx$

d) $\int_a^d f(x)dx$ e) $\int_0^d f(x)dx$ f) $\left|\int_0^d f(x)dx\right|$

g) $\int_0^d |f(x)|dx$ [*Hint*: *Consider the graph of* $y = |f(x)|$.

8. Suppose that after t seconds, a particle moving on a horizontal line has velocity $v(t)$ meters per second (where the positive direction is taken to be *right*). The graph of this velocity function is shown in the figure at right.



Explain the physical significance of each of the following three integrals:

a) $\int_0^2 v(t)dt$ b) $\int_0^5 v(t)dt$ c) $\int_0^5 |v(t)|dt$.

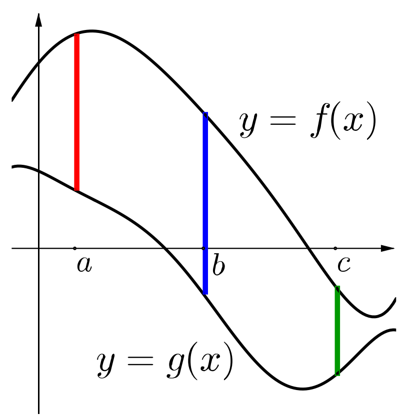
Now decide whether each of the following are true or false:

d) $\int_0^2 v(t)dt = \int_0^2 |v(t)|dt$. e) After five seconds, the particle is to the right of its initial position.

9. Find the numerical values of the following integrals by interpreting them in terms of areas. Draw pictures!

a) $\int_0^1 x\,dx$ b) $\int_{-1}^2 2x\,dx$ c) $\int_{-1}^1 \sqrt{1-x^2}\,dx$ d) $\int_0^2 -\sqrt{4-x^2}\,dx$ e) $\int_2^5 dx$ [Hint: $dx = 1dx$.]

10. a) Write expressions for the areas of each of the three infinitesimally thin rectangles depicted in the figure at right. Note: Even though all three expressions will turn out essentially the same, each case will require a slightly different justification. [*Hint: Bear in mind that while the values of functions may be negative, lengths are always positive*.]



b) In the figure at right, consider the region that lies below $y = f(x)$, above $y = g(x)$, and between the vertical lines $x = a$ and $x = c$. Can this region's area be represented as a single integral? If so, write down the integral that does the job. If not, explain why this can't be done.

c) Sketch the region between the graphs of $y = -x$ and $y = 1-(x-1)^2$ and express its area as an integral.

11. We say a function f is *even* if, for each x in its domain, $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$ for each x.

a) Give some typical examples of even functions and odd functions, including trigonometric examples.

b) The algebraic definitions of even and odd functions have geometric consequences: The graphs of all even functions must exhibit a certain form of symmetry; the graphs of odd functions must exhibit another form. Explain the two forms of symmetry, and explain why they follow from the algebraic definitions.

c) Explain why the following is true: If f is even, then for any a in its domain, $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

d) Explain why the following is true: If f is odd, then for any a in its domain, $\int_{-a}^a f(x)dx = 0$.

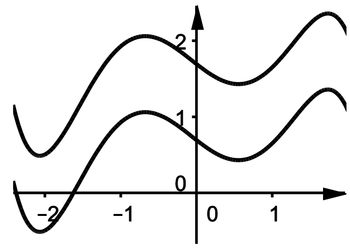
e) Is the function $f(x) = x^3\sin^2 x - x\cos x$ even, odd, or neither?

f) Integrate: $\int_{-\pi}^{\pi}(x^3\sin^2 x - x\cos x)\,dx$.

Preparation for the FTC: The Antiderivative Lemma

An **antiderivative** of a function f is a function whose derivative is f. (One antiderivative of $2x$, for example, is x^2. Another is x^2+1.)

Suppose we have two different antiderivatives of the same function. How different can they be? Well, by definition, their derivatives are equal, so their graphs change at *equal rates* throughout their common domain. When the graph of one increases rapidly, so does the graph of the other; when the graph of one decreases slowly, so does the graph of the other. This perfectly synchronized "dance of the antiderivatives" ensures that the distance between their graphs remains constant. (For it to change, the two graphs would have to change somewhere at *different* rates.)



The moral of the story is the following lemma:*

Antiderivative Lemma. If g and h have the same domain and are antiderivatives of the same function, then they differ only by a constant. That is, $g(x) = h(x) + C$ for some constant C.

If we know one antiderivative of a function, we know them all: An antiderivative of $\cos x$ is $\sin x$; our lemma then guarantees that *every* antiderivative of cosine has the form $\sin x + C$ for some constant C.

Exercises.

12. An important rule you should immediately memorize: *To find an antiderivative of cx^r, where c is a constant, we increase its exponent by* 1 *and divide by the new exponent*.

a) Verify that an antiderivative of cx^r is $cx^{r+1}/(r+1)$, as claimed.

b) Write down antiderivatives for the following functions: $x^3,\ 2x^9,\ x,\ -5\sqrt{x},\ \sqrt[3]{x^2},\ \frac{1}{x^2},\ \frac{-8}{x^{3/7}},\ x^{-\pi}$.

c) State the one value of r for which this rule does *not* apply. Why doesn't the rule apply in this case?

13. Think of an antiderivative of each of the following functions:

a) $\cos x$ b) $-\sin x$ c) $\sin x$ d) $\sec^2 x$ e) e^x f) $\sec x \tan x$ g) $\dfrac{1}{1+x^2}$ h) $-\csc^2 x$ i) $1/x$

14. You can sometimes find an antiderivative by guessing something close, then adjusting your guess to make its derivative come out right. For example, to find an antiderivative of $\sin(3x)$, we'd guess, "It will be something like $\cos(3x)$. But this function's derivative is $-\mathbf{3}\sin(3x)$, which is off by a constant factor of 3. To compensate, let's try multiplying our prospective *anti*derivative by $-1/3$. Will that work? Yes! A quick check shows that the derivative of $-(1/3)\cos(3x)$ is indeed $\sin(3x)$, so we've found our antiderivative."

Use this guess-and-adjust method to find antiderivatives of the following functions.

a) $\cos(5x)$ b) $\sin(\pi x)$ c) e^{-x} d) e^{3x} e) $\sec^2(x/2)$ f) 10^x g) 5^{-x} h) $\dfrac{1}{1+4x^2}$

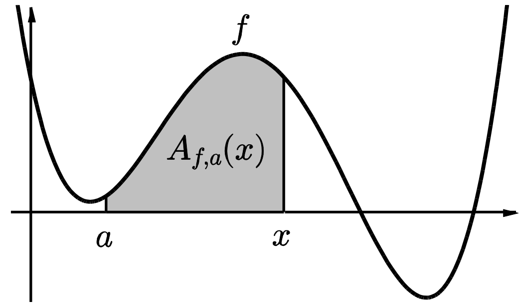
15. Verify that $\ln(x)$ and $\ln(-x)$ are both antiderivatives of $1/x$, and yet they do *not* differ by a constant!

a) Why does this *not* violate our Antiderivative Lemma?

b) Is there an antiderivative of $1/x$ with the same domain as $1/x$? If so, what is it?

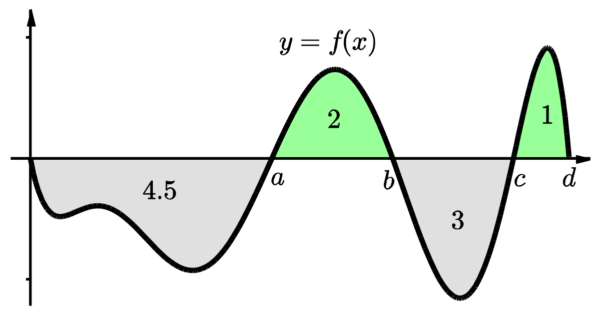
* A "lemma" is a small technical theorem used as a building block in the proof of a bigger, much more important theorem. After introducing one more preliminary idea, we'll use our lemma to establish the Fundamental Theorem of Calculus.

Preparation for the FTC: Accumulation Functions



Take any function f and any fixed point a in its domain. From a, go down the horizontal axis to x, a variable point. In doing so, imagine "accumulating" all the area lying above the interval $[a, x]$ and below f's graph. The amount of area that we accumulate is clearly a function of x. We call this function, not surprisingly, the **accumulation function** $A_{f,a}$. (See the figure at right, which is worth a thousand words.)

If and when f's graph dips below the horizontal axis, the accumulation function *subtracts* any area it obtains from its running total, since any such area lies on the "wrong side" of the axis. Hence, an accumulation function's value will be positive or negative according to whether the majority of the area it accumulates lies above or below the horizontal axis.



Example. Let the function f be defined by the graph in the figure at right. In this case, the following statements hold:

$$A_{f,a}(b) = 2, \quad A_{f,a}(c) = -1,$$
$$A_{f,a}(d) = 0, \quad A_{f,b}(c) = -3,$$
$$A_{f,b}(d) = -2, \quad A_{f,0}(b) = -2.5\,.$$

Once you've digested this example, you are ready to check your understanding further by doing the following exercises.

Exercises.

16. If $g(x) = 4 - x$, evaluate the following accumulation functions at the given inputs:

a) $A_{g,0}(4)$ b) $A_{g,2}(6)$ c) $A_{g,-2}(4)$ d) $A_{g,1}(1)$

17. If $s(x) = \sin x$, find the value of $A_{s,0}(2\pi)$.

18. We can express accumulation functions in integral notation. It is almost, but not quite, correct to write

$$A_{f,a}(x) = \int_a^x f(x)dx.$$

The problem is that this integral purports to sum things up while x runs from a to... x, which is syntactically incoherent. We stop summing when x is x? But when is x *not* x? To restore coherence, we recall that the variable in a function's *formula* is just a placeholder, a "dummy". (For example, it is just as reasonable to write the squaring function's formula as $f(t) = t^2$ as it is to write it as $f(x) = x^2$.) We therefore write

$$A_{f,a}(x) = \int_a^x f(t)dt,$$

which takes care of the problem, since we are now summing things up as $\boldsymbol{t}$ runs from a to x.

Your problem: If $g(x) = \cos x$, express the function $A_{g,0}(x)$ as an integral.

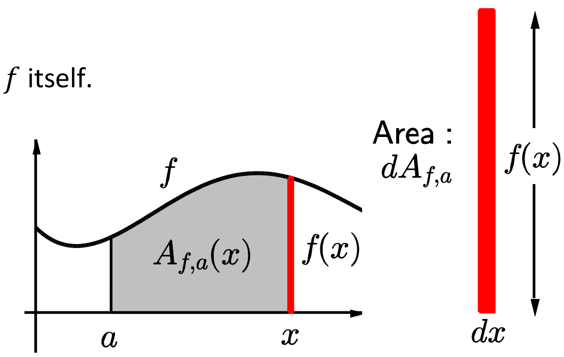
The Fundamental Theorem of Calculus (Stage 1: The Acorn)

"Only by considering infinitesimal units for observation (the differential of history, the individual tendencies of men) and acquiring the art of integrating them (finding the sum of these infinitesimals) can we hope to arrive at laws of history."

–Tolstoy, *War and Peace*, Epilogue, Part 2.

The Fundamental Theorem of Calculus develops in two stages, the first of which contains the second – in the sense that an unremarkable-looking acorn contains, *in potentia*, a mighty oak.

FTC (Acorn Version).
The derivative of any accumulation function of f is f itself.



Proof. Let $A_{f,a}$ be an accumulation function for f. We'll find its derivative by the usual geometric procedure: We'll increase x by an infinitesimal amount dx, note the corresponding infinitesimal change $dA_{f,a}$ in the function's value, and finally, take the ratio of these two changes.

Since the accumulation function's value is the shaded area under the graph, its infinitesimal increase $dA_{f,a}$ (obtained by nudging x forward by dx) is the area of the infinitesimal rectangle I've emphasized in the figure. Dividing this rectangle's area by its width yields its height, $f(x)$. Rewriting this last sentence in symbols, we have

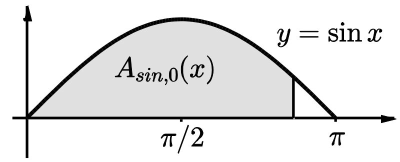
$$\frac{dA_{f,a}}{dx} = f(x).$$

This is exactly what we wanted to prove: The derivative of f's accumulation function is f itself. ■

The proof is simple enough, but having digested it, you probably remain puzzled by the result itself. After my grand promises that the FTC will help us evaluate integrals, I've presented you instead with a strange little acorn. What is it? What is it trying to tell us? Let us look closer.

Since an accumulation function is effectively an integral (a point made explicit in exercise 18), the acorn version of the FTC suggests that integration and differentiation are *inverse* processes. That is, if we start with a function f, take its integral (i.e. form an accumulation function from it), and then take the derivative of the result, we end up right back where we started – with f itself. Considering it this way, we recognize the acorn's latent power: It establishes a link between the well-mapped terrain of derivatives and the still-mysterious land of integrals. Just as we can use the simple properties of exponentiation to establish the basic laws of its inverse process (taking logarithms), we will now be able to use the simple properties of derivatives to crack the code of integrals.

To grow into an oak, our acorn still needs nutrients from the Antiderivative Lemma. In the following example, we'll bring the lemma and the acorn together at last. (As a fun but unimportant bonus, we'll also establish a surprising result about the sine wave.)



Example. Find the area under one arch of the sine wave.

Solution. The area we want is $A_{sin,0}(\pi)$. To find this value, we'll first produce an explicit formula for the accumulation function $A_{sin,0}(x)$, and then we'll substitute π into it.

To find our formula, we first note that $A_{sin,0}(x)$ is an antiderivative of $\sin x$ (by the FTC Acorn). Then, since we happen to know a formula for another antiderivative of $\sin x$ (namely, $-\cos x$), our Antiderivative Lemma guarantees that $A_{sin,0}(x) = -\cos x + C$ for some constant C. A-ha!

To find C's value, we substitute zero (our accumulation function's starting point) for x in the previous equation. Doing so shows that C must be 1, as you should verify. Hence, we now have

$$A_{sin,0}(x) = -\cos x + 1,$$

from which it follows that the area under one arch of the sine wave is exactly

$$A_{sin,0}(\pi) = -\cos \pi + 1 = 2! \quad \blacklozenge$$

That the area is a whole number is remarkable. Much more remarkable, however, is the argument by which we obtained this result. You'll use this argument in the exercises below. In the next section, we'll generalize it to produce the full FTC in all its oaky glory.

Exercises.

19. Sketch the region that lies below the graph of $y = (1/3)x^2$, above the x-axis, and between $x = 0$ and $x = 2$. Then, using the acorn version of the FTC (as in the example above), find its area.

20. Do the same for the region lying below $y = 1/(1 + x^2)$, above the x-axis, and between $x = 0$ and $x = \sqrt{3}$. [*Hint for the graph: Start with the graph of $y = 1 + x^2$. Think of how it would change if all of its points' y-coordinates were changed to their reciprocals; points far from the horizontal axis would be brought close to it, and vice-versa.*]

21. The acorn version of the FTC is often stated as follows: $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$.

a) Convince yourself that this symbolic statement is equivalent to the verbal statement of the Acorn FTC I've given above. (Recall exercise 18.)

b) Find the following derivatives: $\frac{d}{dx}\left(\int_a^x \sin(t)dt\right)$, $\frac{d}{dx}\left(\int_a^x \sin(t^2)dt\right)$, $\frac{d}{dx}\left(\int_a^{x^2} \sin(t)dt\right)$.

[*Hint for the third one: Let y be the function whose derivative you seek. Let $u = x^2$.* Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.]

22. Strictly speaking, our geometric proof of the Acorn FTC is not quite complete, though its holes are tiny and easily patched; you'll provide the necessary spackle in this problem.

a) The problem is that our proof tacitly assumed that $f(x) > 0$ at the point x where we took the accumulation function's derivative. Pinpoint the exact sentence in the proof in which we first used this assumption.

b) Having identified the flaw, we must show that the Acorn FTC holds even if we take the derivative at a point where $f(x) < 0$. To do so, draw a picture representing this situation. Since nudging x forward by dx will produce, in this case, an infinitesimal *decrease* in the accumulation function, we know that $dA_{f,a}$ will be the *negative* of the area of the rectangle in your picture. Translate this last statement into an equation; from it, deduce that even in this case, $dA_{f,a}/dx = f(x)$.

c) What if $f(x) = 0$ at the point where we take the accumulation function's derivative?

The Fundamental Theorem of Calculus (Stage 2: The Oak)

Once upon a time, you learned a clever technique for solving quadratics: completing the square. After using this technique a few times to solve particular quadratics, you applied it in the abstract to the equation $ax^2 + bx + c = 0$, thereby deriving the quadratic formula. With this formula in hand, you never again needed to complete the square to solve individual quadratics; the formula completes the square for you under the hood, automating the process so that you don't have to think about it.

Something similar is about to happen here. Having used the Acorn FTC to evaluate a few specific integrals, we'll now apply it to an integral in the abstract, thereby deriving the full-grown FTC, which we'll use thereafter to evaluate any integrals we meet, content to let it automate the details for us. We'll also drop the distinction between the two stages of the FTC; the oak contains the acorn as surely as the acorn the oak.

Recall the previous section's clever argument. We can think of an integral as representing one particular value of an accumulation function. By the Acorn FTC, this accumulation function is an *antiderivative* of the function being integrated. By the Antiderivative Lemma, we can express this accumulation function in terms of any other known antiderivative of the function being integrated. Consequently, *if we know another antiderivative* of the function being integrated, then we can find a formula for the accumulation function, which in turn allows us to evaluate the integral. It is a remarkable argument, and one that cries out for automation. Let us hearken unto its cries.

The FTC. If f is continuous over $[a, b]$ and has an antiderivative F, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Using the notation for accumulation functions introduced two sections ago, we note that the integral is equal to $A_{f,a}(b)$. To evaluate this expression (and thus to evaluate the integral), we'll find a formula for $A_{f,a}(x)$, and then let $x = b$.

By the Acorn FTC, we know that $A_{f,a}(x)$ is an antiderivative of f. Since F is an antiderivative as well, the Antiderivative Lemma assures us that $A_{f,a}(x) = F(x) + C$ for some constant C.

To find C's value, we let x be a in the preceding equation. Doing so yields $C = -F(a)$, so the equation in the preceding paragraph can be rewritten as $A_{f,a}(x) = F(x) - F(a)$. Consequently,

$$\int_a^b f(x)dx = A_{f,a}(b) = F(b) - F(a),$$

as claimed. ■

The Fundamental Theorem of Calculus tells us that when we integrate a function over an interval, the result depends entirely on the values of the function's antiderivative at the interval's two endpoints. This is – or at least it should be – astonishing: The interval consists of infinitely many points, and the integral is a sum of infinitely many terms… yet somehow the integral's value depends only on the antiderivative's value at *two* points? How can this be? I encourage you to sit by the fire some winter evening and meditate upon this mystery, bearing it in mind while thinking your way through the chain of proofs that led us to the Fundamental Theorem. The theme of understanding a function's behavior throughout a region by understanding a related function's behavior on the region's *boundary* will return when you study vector calculus; there, the regions you'll consider will be not just intervals of the one-dimensional real line, but regions of two, three, or higher-dimensional spaces.

From a pragmatic perspective, the FTC is of supreme importance because it lets us evaluate any integral once we know an antiderivative of the function being integrated. We simply evaluate the function's antiderivative at the boundaries of integration and subtract.

Example 1. Evaluate the integral $\int_2^3 x^4 dx$.

Solution. An antiderivative of x^4 is $F(x) = x^5/5$, so by the FTC,

$$\int_2^3 x^4 dx = F(3) - F(2) = \frac{243}{5} - \frac{32}{5} = \frac{211}{5}. \blacklozenge$$

To simplify our written work when evaluating integrals, some special notation has been developed:

$$\left[\boldsymbol{F(x)}\right]_a^b \textbf{ is shorthand for } \boldsymbol{F(b) - F(a)}.^*$$

This handy bracket notation lets us dispense with explanatory phrases such as "Since $F(x)$ is an antiderivative of blah-blah-blah, the FTC tells us that…" when evaluating an integral. If, for instance, we redo the previous example with this notation, we can reduce the entire solution to just a few symbols:

Example 1 (Encore). Evaluate the integral $\int_2^3 x^4 dx$.

Solution. $\int_2^3 x^4 dx = \left[x^5/5\right]_2^3 = \frac{243}{5} - \frac{32}{5} = \frac{211}{5}. \quad \blacklozenge$

This exemplifies the usual written pattern for using the FTC to evaluate simple integrals: We write down an antiderivative along with the attendant bracket notation, then we evaluate. Here's another quick example to ensure that the pattern is clear.

Example 2. Evaluate the integral $\int_{1/2}^1 \frac{2}{\sqrt{1-x^2}}\, dx$.

Solution. $\int_{1/2}^1 \frac{2}{\sqrt{1-x^2}}\, dx = \left[2 \arcsin x\right]_{1/2}^1 = 2\arcsin 1 - 2\arcsin(1/2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}. \quad \blacklozenge$

Polynomials will be especially easy to integrate once we've made two small observations. We've made the first already (in exercise 10): To find an antiderivative of cx^n, we raise the exponent by one and divide by the new exponent. The second is that *antiderivatives, like derivatives, can be taken term by term*.† Thus, to find an antiderivative of $x^3 + 2x^2 - 3x$, we simply sum up the terms' antiderivatives to obtain $(x^4/4) + (2x^3/3) - (3x^2/2)$. Armed with these observations, integrating a polynomial is trivial.

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* The left bracket is often omitted by those for whom laziness trumps symmetry.

† Antiderivatives can be taken term by term *because* derivatives can. In symbols, if F and G are antiderivatives of f and g, then an antiderivative of $\big(f(x) + g(x)\big)$ is $\big(F(x) + G(x)\big)$ *because* $\big(F(x) + G(x)\big)' = F'(x) + G'(x) = f(x) + g(x)$.

Example 3. Evaluate the integral $\int_0^2 (-2x^2 + 4x + 1)dx$.

Solution. $\int_0^2 (-2x^2 + 4x + 1)dx = \left[-\frac{2}{3}x^3 + 2x^2 + x\right]_0^2 = \left(-\frac{16}{3} + 8 + 2\right) - 0 = \frac{14}{3}$. ♦

Exercises.

23. Evaluate the following integrals, making use of the bracket notation introduced above.

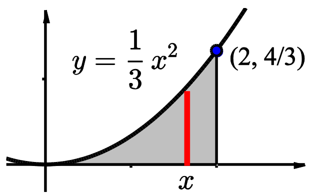
a) $\int_0^{3\pi/2} \cos x \, dx$ b) $\int_1^2 -2x^3 dx$ c) $\int_1^{e^2} \frac{1}{x} dx$ d) $\int_1^{\sqrt{3}} \frac{1}{1+x^2} dx$ e) $\int_{\pi/6}^{\pi/4} \sin x \, dx$ f) $\int_{-1}^{8} \sqrt[3]{x} \, dx$

g) $\int_0^{\pi/9} \cos(3x) \, dx$ [*Hint: Recall exercise* 14.] h) $\int_0^{\sqrt{3}/6} \frac{1}{\sqrt{1-9x^2}} dx$ i) $\int_0^1 (x^9 + 5x^4 - 3x)dx$

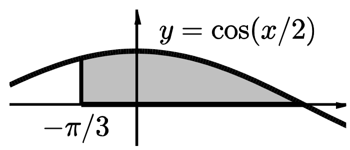
j) $\int_0^{\pi/8} 2 \sin x \cos x \, dx$ [*Hint: Use a trigonometric identity to rewrite the function being integrated*.]

24. Find the areas of the shaded regions shown below:

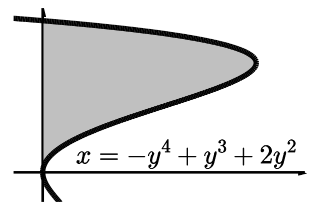
a)



b)



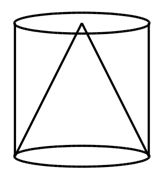
c)



25. Consider the region that lies in the first quadrant, under $y = \sqrt[3]{x}$, and to the left of $x = 8$. By revolving it about the x-axis (as in exercise 4), we obtain a solid of revolution. Sketch it, mentally decompose it into infinitesimally thin slices, then draw a typical slice and find its volume. Finally, find the volume of the full solid.

26. Consider the region lying in the first quadrant, above $y = \sqrt[3]{x}$, and below $y = 2$. Revolve it around the $\boldsymbol{y}$-axis. Sketch it and so forth, ultimately finding its volume. [*Hint: To find the radius of a typical slice, you'll want to think of x as a function of y, rather than the other way around*.]

27. You probably know Democritus's famous theorem about cones: A cone takes up exactly $1/3$ of the space of the cylinder containing it. (See the figure.) Since a cylinder's volume is, of course, its base's area times its height, Democritus's result allows us to write down a formula for a cone's volume in terms of its base radius and height: $V_{cone} = (1/3)\pi r^2 h$. Please do not clutter your memory with this formula; rather, just remember Democritus's result directly, from which you can reconstruct the formula in seconds whenever you need it.



a) Prove Democritus right. [*Hint: Think of the cone as a solid of revolution generated by a line segment with one endpoint at the origin. Find the equation of the line containing the segment; naturally, it will involve r and h. Use calculus to find the generated cone's volume and verify that it agrees with the formula above*.]

b) Learn about Democritus, the laughing atomist philosopher.

28. Here is a sometimes useful fact: **Swapping an integral's boundaries of integration multiplies its value by -1.*** To see why this is so, return to first principles: To integrate is to sum up $f(x)dx$'s. If we integrate *backwards* (from right to left on the number line), we'll encounter the same values of $f(x)$ that we meet when integrating in the usual way, but the dx's – the infinitesimal changes in x – will be *negative*, since x *decreases* as we move leftwards on the number line. Thus, each $f(x)dx$ in the sum will be the negative of what it would have been had we integrated in the usual direction. The net result is that the value of our "backwards" integral is precisely the negative of its "forward" counterpart. *Your problem: Meditate upon this until it makes sense.*

* For example, since $\int_0^{\pi} \sin x \, dx = 2$, our "useful fact" immediately tells us that $\int_{\pi}^{0} \sin x \, dx = -2$.

29. Rewrite the following as a single integral: $\int_a^b f(x)dx - \int_c^b f(x)dx + \int_c^d f(x)dx$. [*Hint: Exercise* 28 will help.]

30. Given the function f in exercise #7, evaluate the following integrals:

a) $\int_b^a f(x)dx$ b) $\int_c^b f(x)dx$ c) $\int_c^0 f(x)dx$.

31. a) Prove that a constant factor c can be pulled through the bracket notation. That is, $[cF(x)]_a^b = c[F(x)]_a^b$.

b) Use the fact that you can pull a constant multiple through the bracket while evaluating these integrals:

$\int_0^{3\pi/10} \cos(5x)dx$, $\int_0^1 e^{-5x}\,dx$, $\int_4^{25} \sqrt{x}\,dx$, $\int_0^{1/4} \frac{1}{1+16x^2}\,dx$

32. (The linearity properties of the integral)

a) Convince yourself that **constant factors can be pulled through integrals**.

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \text{ for any constant } c.$$

[*Hint: You can "see" this fact geometrically by thinking about rectangles' areas. Or think arithmetically: An integral is a sum; you should be able to convince yourself that the property above is essentially a matter of pulling out a common factor from each term in the sum.*]

b) Convince yourself that **the integral of a sum is the sum of the integrals**.

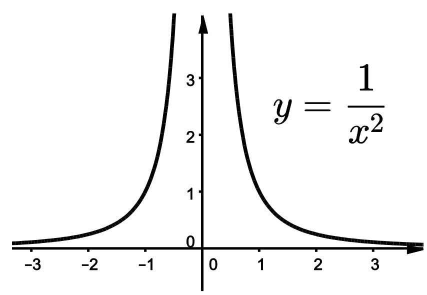
$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

[*Hint: As in the previous part, you can "see" this fact geometrically by thinking about rectangles' areas. Or, think arithmetically: An integral is a sum; you should be able to convince yourself that the property above is essentially a matter of reordering the sum's infinitely many terms.**]

33. Is the integral of a *product* the product of the integrals? If so, prove it. If not, provide a counterexample.

34. A warning: *Vertical asymptotes cause the FTC to fail*. Should you try to integrate over an interval containing one, the result will be gibberish. For example, there is an infinite amount of area below the graph of $y = 1/x^2$ and between $x = -1$ and 1, but if we tried to apply the FTC to the integral $\int_{-1}^1 1/x^2\,dx$, we'd obtain this wholly erroneous "result":

$$\int_{-1}^1 1/x^2\,dx = [-1/x]_{-1}^1 = -2.$$



Your problem: In the statement of the FTC in the text, a phrase warns you that the FTC does *not* apply to this integral. Identify the phrase.

* (A footnote you may safely ignore if you wish.) This innocent-looking statement may induce fits of violent swearing in tetchy calculus teachers. In Chapter 7 you'll learn why: Infinite sums don't always behave as innocently as their finite brethren; under some circumstances, reordering an infinite sum's terms can alter the sum's value. Fortunately, this disturbing phenomenon happens only in infinite sums of a sort never occurring in ordinary integrals, so it need not concern you here.

Still, if you are concerned (or are a tetchy calculus teacher yourself), you may prefer the following proof of the property in question: Let F and G be antiderivatives of f and g respectively. A linearity property of *derivatives* ensures that $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$. Hence, the FTC guarantees that

$$\begin{aligned}\int_a^b \big(f(x)+g(x)\big)dx &= \big[F(x)+G(x)\big]_a^b = \big(F(b)+G(b)\big) - \big(F(a)+G(a)\big)\\ &= \big(F(b)-F(a)\big) + \big(G(b)-G(a)\big) = \big[F(x)\big]_a^b + \big[G(x)\big]_a^b = \int_a^b f(x)dx + \int_a^b g(x)dx.\end{aligned}$$

The preceding argument has the merit of demonstrating how to prove integral properties with the FTC by linking them to related derivative properties, but it conveys no real insight as to *why* the theorem holds. In contrast, "the theorem holds because it's just a rearrangement of the sum's terms" is both insightful and intuitive (after some initial thought), even if lacking in mathematical rigor. The goals at this stage should be insight and intuition; a healthy, astringent dose of rigor can always be added later. Rigor, injected too early, begets rigor mortis.